Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 5, 2004*
Accepted: *July 9, 2004*

A new deconstruction of little string theory

Nick Dorey

Department of Physics, University of Wales Swansea
Singleton Park, Swansea, SA2 8PP, U.K.
E-mail: n.dorey@swansea.ac.uk

Abstract: We present evidence for a new deconstruction of Little String Theory (LST). The starting point is a four-dimensional conformal field theory on its Higgs branch which provides a lattice regularization of six-dimensional gauge theory. We argue that the corresponding continuum limit is a 't Hooft large-N limit of the same four-dimensional theory on an S-dual confining branch. The AdS/CFT correspondence is then used to study this limit in a controlled way. We find that the limit yields LST compactified to four dimensions on a torus of fixed size. The limiting theory also contains other massive and massless states which are completely decoupled. The proposal can be adapted to deconstruct Double-Scaled Little String Theory and provides the first example of a large-N confining gauge theory in four dimensions with a fully tractable string theory dual.

Keywords: Field Theories in Higher Dimensions, 1/N Expansion, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction and overview

One of the more surprising outcomes of recent developments in string theory is the discovery of Lorentz invariant interacting quantum theories without gravity in spacetimes of dimension greater than four. In this paper we will focus on a six-dimensional theory known as Little String Theory (LST) which arises on the world volume of coincident IIB NS5 branes in a certain decoupling limit [1] (for a review see [2, 3]). The theory is non-local but reduces to a conventional six-dimensional non-abelian gauge theory at low energies. The LST corresponding to m NS5 branes has low energy gauge group SU(m). LST is interesting for a number of reasons including its relation to string theory on singular spacetimes and possible phenomenological applications. After compactification, LST also has an interesting relationship to confining gauge theories in four-dimensions [4]. In this paper, we will find a new and precise form of this relationship which implies that LST is fully equivalent to a particular large-N confining gauge theory.

Our approach to understanding LST will be based on the idea of *deconstruction* [5] (For related work see [6]). Deconstruction provides an attractive way of obtaining higher dimensional theories as special limits of more familiar four-dimensional gauge theories. A deconstruction of LST using the large-n limit of a four-dimensional quiver model with gauge group $\mathrm{SU}(m)^{n^2}$ was suggested in [7]. In this paper we will discuss a related proposal [8, 9] based on a different four-dimensional theory with gauge group $\mathrm{U}(mn)$. In either case, the four-dimensional theory has a Higgs branch where the large-n classical spectrum of massive W-bosons coincides with the Kaluza-Klein spectrum of a six-dimensional theory compactified on a torus. In fact the classical theory is equivalent to a lattice regularisation of the six-dimensional $\mathrm{SU}(m)$ gauge theory which arises as the IR limit of LST. In both cases the proposal is that LST itself can be obtained as a continuum limit of this lattice theory. The $\mathrm{U}(mn)$ construction of [8, 9] actually yields a non-commutative generalization of lattice gauge theory. However, we will see that the theory becomes commutative in the continuum limit.

The emergence of a six dimensional lattice theory observed in [7, 8, 9] is based on classical arguments which are only valid at weak coupling. However, as we review below, obtaining a continuum theory necessarily involves taking a large-n, strong-coupling limit. The plausibility of deconstruction (and its usefulness) are dependent on understanding this limit. In this paper we will use field theory and string theory methods to study the continuum limit of [9] in a controlled way. As suggested in [9], the first step is to use S-duality to reinterpret the continuum limit as a more or less conventional 't Hooft limit of a confining gauge theory. We will then use the AdS/CFT correspondence to construct a holographic dual of the strongly coupled confining theory. The RG flow from four dimensional behaviour in the UV to a six-dimensional theory in the IR can then be exhibited directly. We are able to show that the proposed continuum limit does indeed yield LST, although the details are quite different from weak-coupling expectations. As an application of these ideas, we adapt our proposal to deconstruct double-scaled Little String Theory (DSLST). Weak coupling calculations in DSLST provide exact results for the large-N glueball spectrum of the dual gauge theory. In the remainder of this introductory section we will give an overview of the main results. The details are fleshed out in the remaining sections.

In the version of deconstruction suggested in [8, 9], the appearance of additional space-time dimensions follows from a phenomenon which is very familiar in the context of M(atrix) theory [11]. We will start from $\mathcal{N}=4$ SUSY Yang-Mills in four-dimensions with gauge group $\mathrm{U}(N)$ (for $N=mn$) as realized on the world volume of N D3 branes in type-IIB string theory. The $\mathcal{N}=4$ theory contains three complex adjoint scalar fields denoted Φ_1, Φ_2 and Φ_3. We will choose a non-zero background for these fields obeying,

$$\Phi_1\Phi_2 = \exp\left(-\frac{2\pi i}{n}\right)\Phi_2\Phi_1\,, \tag{1.1}$$

and then expand the fields around this background. In a particular $n \to \infty$ limit, the resulting theory is classically equivalent to a six-dimensional U(m) gauge theory compactified on a torus [33, 34]. In string theory, this is interpreted as the polarization of N D3 branes into m D5 branes wrapped on a two-dimensional torus in the transverse dimensions. At the classical level, similar considerations apply to other theories with sixteen supercharges including the matrix quantum mechanics obtained by reduction of the $\mathcal{N}=4$ theory to $0+1$ dimensions. The latter case leads to the construction of toroidally wrapped memebranes from D0 branes in M(atrix) theory [12, 13].

One potential problem with this proceedure is that the solutions of (1.1) are not vacuum states of the $\mathcal{N}=4$ theory and so the corresponding D3/D5 configuration is unstable. However, this is easily remedied by deforming the $\mathcal{N}=4$ theory. In particular, we can consider instead the theory with superpotential,

$$\mathcal{W} = \mathrm{Tr}_N \left[e^{i\frac{\beta}{2}} \Phi_1 \Phi_2 \Phi_3 - e^{-i\frac{\beta}{2}} \Phi_1 \Phi_3 \Phi_2 \right]. \tag{1.2}$$

Here the three complex scalars of the $\mathcal{N}=4$ theory have been promoted to $\mathcal{N}=1$ chiral superfields. In the $\mathcal{N}=1$ language, the theory also contains a U(N) vector multiplet. As in [9], we will refer to this model as the β-deformed theory. The undeformed $\mathcal{N}=4$ theory corresponds to $\beta=0$. For $\beta=2\pi/n$, one of the resulting F-term equations coincides with our background condition (1.1). The solutions of (1.1) therefore yield stable vacua which preserve the $\mathcal{N}=1$ supersymmetry of the deformed theory. The non-trivial solutions of (1.1) correspond to a Higgs branch of the theory where U(N) is broken down to a U(m) subgroup.

In the string theory set-up, the deformation (1.2) of the $\mathcal{N}=4$ superpotential corresponds to a particular background value for the Ramond-Ramond three-form field strength. The effect of the background field is to stablize the corresponding configuration of toroidally-wrapped D5 branes. This is entirely analogous to the Myers effect [14] which causes the polarization of D3 branes into spherically-wrapped D5 branes in the string theory dual of the $\mathcal{N}=1^*$ SUSY Yang-Mills studied by Polchinski and Strassler [15]. Many features of the analysis given in this paper run parallel to the $\mathcal{N}=1^*$ case of [15] although there are also some important differences.

The appearance of extra dimensions in the U(mn) β-deformed theory is also closely related to the conventional set-up for deconstruction based on a quiver theory with gauge group SU$(m)^{n^2}$. In both cases the theory has a Higgs branch with unbroken gauge group U(m) (or SU(m)) and the large-n spectrum of massive W-bosons on the Higgs branch provides two towers of Kaluza-Klein (KK) states. At finite-n, each KK tower is truncated in a way which corresponds to a discretization of the additional dimensions. In both cases the full classical action can be interpreted as a discretised version of six-dimensional gauge theory, defined on $R^{3,1} \times \mathcal{L}$ where $\mathcal{L}$ is an $n \times n$ lattice with periodic boundary conditions. A key difference is that expanding the U(mn) theory around the background (1.1) yields a *non-commutative* lattice gauge theory [8, 10]. In contrast the conventional approach to deconstruction based on gauge group SU$(m)^{n^2}$ yields an ordinary commutative lattice theory.

Although the two approaches to deconstruction start from theories with very different gauge group and matter content, string theory provides an easy way to understand the relation between them [8]. The quiver theory of [7] is realized in IIB string theory as the worldvolume theory of m D3 branes placed at a $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold singularity. On the other hand, the β-deformed theory with gauge group $\mathrm{U}(mn)$ and $\beta = 2\pi/n$ can be realised in string theory in (at least) two ways. As above we can consider mn D3 branes with non-zero RR three-form background. An alternative construction of the same theory is to place m D3 branes at a $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold singularity with a single unit of *discrete torsion* [16]. As in the quiver construction of [7], the truncated towers of KK states correspond to fundamental strings stretched between D3 branes and their image points under the orbifold group. At large-N, the orbifold becomes a sharp cone over an $S^5/\mathbb{Z}_n \times \mathbb{Z}_n$ base. In an appropriate scaling limit neighborhood of a point on the cone becomes $R^4 \times T^2$ and, after T-dualizing both compact directions, the m D3 branes become m toroidally wrapped D5 branes [7]. In the $\mathrm{U}(mn)$ construction, the limit also converts the discrete torsion into a background value for B_{NS} on T^2 which induces world-volume non-commutativity on the wrapped D5 branes [8].

In this paper we will focus on the deconstruction of six-dimensional gauge theory provided by the β-deformed $\mathrm{U}(mn)$ theory described above. In the following g^2 denotes the four-dimensional $\mathrm{U}(mn)$ gauge coupling. It is instructive to relate the parameters of the resulting lattice theory to those of the underlying four-dimensional gauge theory. On the Higgs branch the gauge symmetry is broken from $\mathrm{U}(mn)$ down to $\mathrm{U}(m)$ at a scale[1] v set by the VEVs. The six-dimensional theory is then characterised by the following length scales,

$$G_6 \sim \sqrt{g^2 n}\,\mathrm{v}^{-1} \qquad \varepsilon \sim \mathrm{v}^{-1} \qquad \sqrt{\theta} \sim \sqrt{n}\,\mathrm{v}^{-1} \qquad R \sim n\,\mathrm{v}^{-1}\,. \tag{1.3}$$

Here G_6 is the six-dimensional gauge coupling, ε is the lattice spacing, $\sqrt{\theta}$ is the length-scale of non-commutativity and R is the radius of the compact dimensions. For weak coupling and large-n, the four length-scales given in (1.3) appear in ascending order and are well seperated. As we go from small length scales to larger ones, the classical theory undergoes RG flow from a four-dimensional conformal field theory in the UV ($l \ll \varepsilon$), to a six dimensional non-commutative gauge theory ($\varepsilon \ll l \ll R$) and finally to an ordinary four-dimensional gauge theory in the far IR ($R \ll l$).

The fact that the lattice spacing is much larger than the six-dimensional gauge coupling is a characteristic feature of weak-coupling deconstruction and indicates that the lattice theory is far from the continuum. Ideally we would like to find a continuum limit in which the non-commutivity scale $\sqrt{\theta}$ and lattice spacing ε go to zero with G_6 and R held fixed. Extrapolating the weak coupling formulae in (1.3) indicates that this can achieved taking $n \to \infty$ with $g^2 \sim n$ and $\mathrm{v} \sim n$. Naively this should yield a commutative theory with $\mathcal{N} = (1,1)$ super-Poincare invariance in six dimensions compactified to four dimensions on a torus of fixed size. This continuum theory should reduce to a $\mathrm{U}(m)$ gauge theory at low energies. As LST is the only known theory with these properties it is the natural candidate

[1] In fact there can be several independent scales set by the VEVs of the different scalar fields but for simplicity we will supress this. For full details see section 3 below.

for the theory which arises in our proposed continuum limit. However, to understand this limit properly we certainly need to understand the quantum corrections to the classical picture of deconstruction described above. For example, one could easily imagine that the classical formula for ε is corrected in such a way that the lattice spacing never vanishes.

The preceeding discussion indicates that the interesting questions about the existence and nature of a continuum limit are hidden in the strongly-coupled dynamics of the four-dimensional gauge theory. Several remarkable properties of the quantum theory with superpotential (1.2) will allow us to make progress in answering these questions. For other relevant work on this model see [18]. The first point is that the parameter β corresponds to an *exactly marginal* deformation of the $\mathcal{N}=4$ theory [17]. The deformation parameter β does not run and therefore parametrizes a family of $\mathcal{N}=1$ superconformal field theories. The Higgs branch which appears classically for $\beta=2\pi/n$ persists in the quantum theory for all values of the gauge coupling [9]. Another remarkable property, uncovered in [9, 19], is that the the theory has an exact electric-magnetic duality extending that of the $\mathcal{N}=4$ theory. As in the $\mathcal{N}=4$ theory the duality inverts the coupling constant: $g^2 \to \tilde{g}^2 = 16\pi^2/g^2$. The transformation also acts non-trivially on the deformation parameter taking the theory with $\beta = 2\pi/n$ described above to a dual theory with $\tilde{\beta} = 8\pi i/\tilde{g}^2 n$. The theory with this imaginary value of the deformation parameter has no Higgs branch but instead has a new branch, invisible classically, where the $\mathrm{U}(N)$ gauge symmetry is confined down to $\mathrm{U}(m)$ at a scale[2] $\tilde{\mathrm{v}} = \tilde{g}^2 v/4\pi$. The physics of this phase is discussed in [9]. The candidate continuum limit of the Higgs branch theory can now be reinterpreted as a limit of the theory on this new quantum confining branch. Specifically we must take now take the limit $n \to \infty$, $\tilde{g}^2 \to 0$ with $\tilde{g}^2 n$ and $\tilde{\mathrm{v}}$ held fixed.

Interestingly the S-dual continuum limit is something quite familiar: a 't Hooft large-N limit[3] of gauge theory in a confining phase, although the confinement is only partial. This is a limit in which we expect Yang-Mills theory to exhibit string-like behaviour. On the other hand LST, which we will claim arises in this limit, is a non-critical theory of closed strings. It is natural to suspect that the Little String is one and the same as the confining string in large-n gauge theory. In the final section of the paper, we will find a region of parameter space where this correspondence can be made quite precise.

As promised above our main goal is to find a weakly-coupled dual description in which we can study the proposed continuum limit. We will accomplish this by applying yet another duality to the confining phase theory. In particular, when $\tilde{g}^2 n \gg 1$, we have $|\tilde{\beta}| = 8\pi/\tilde{g}^2 n \ll 1$. The resulting theory is then a small deformation of $\mathcal{N}=4$ SUSY Yang-Mills at large-N, with large 't Hooft coupling $\tilde{g}^2 N \gg 1$. The conformally invariant vacuum of the theory therefore has a reliable description in IIB supergravity as a small deformation of $AdS_5 \times S^5$ [21]. The deformation in question involves the introduction of non-zero NS three-form flux on the boundary of AdS_5 and has been worked out explicitly in [22, 23].

[2]As explained in [9], the factor of $\tilde{g}^2/4\pi$ in the relation between the scales v and $\tilde{\mathrm{v}}$ comes from the transformation properties of chiral operators under S-duality.

[3]Throughout this paper we have $N = mn$ and the large-N limit we consider corresponds to taking $n \to \infty$ with m fixed.

JHEP07(2004)016

Following the ideas of Polchinski and Strassler [15], we can also find AdS duals for the various Higgs and confining phase ground states of this theory by introducing wrapped fivebranes embedded in this geometry. In particular, the string dual of the confining vacuum involves m NS fivebranes wrapped on a two-dimensional torus $T^2 \subset S^5$. The fivebranes are located at fixed radial distance. The N D3 branes also expand to lie on the same toroidal shell and the resulting geometry is warped accordingly As in the $\mathcal{N} = 1^*$ case, we will find an approximate supergravity solution (valid for $\tilde{g}^2 n \gg m$) corresponding to this brane configuration.

Far from the branes, the dual geometry of the confining phase asymptotes to $AdS_5 \times S^5$ deformed by a background NS threeform flux. This corresponds to the strongly-coupled four-dimensional superconformal fixed point which controls the UV behaviour of the dual field theory. As we approach the branes, the solution makes a smooth transition to the near-horizon geometry m NS5 branes. The spectrum of theory includes the U(m) gauge fields living on the six-dimensional world volume of the NS5 branes. The SUGRA solution fixes the six-dimensional parameters in terms of the four-dimensional ones. We can then compare these strong coupling results from a naive extrapolation of the classical formulae (1.3). We find that the strong and weak coupling results for the six-dimensional gauge coupling G_6 and the compactification radius R agree. In addition the low energy six-dimensional gauge theory in this regime is commutative as expected.

The dual geometry described above encodes the RG flow from a four dimensional CFT in the UV to a six-dimensional gauge theory in the IR. In particular it determines the mass scale at which this transition takes place. We find that the strongly-coupled theory behaves like a four-dimensional CFT above the scale $\Lambda \sim \tilde{\mathrm{v}}/\sqrt{\tilde{g}^2 mn}$. At weak coupling the corresponding scale is set by the inverse lattice spacing ε^{-1}. Here we find a significant discrepancy between the strong and weak coupling results. In particular, the strong-coupling scale Λ remains fixed in our proposed continuum limit. We conclude that the theory does not recover six-dimensional Lorentz invariance in this limit.

Despite this negative result, the dynamics of the strongly-coupled theory simplifies in an interesting way in the large-n 't Hooft limit discussed above. The supergravity dual of the confining phase involves m NS fivebranes embedded in a geometry which is asymptotically AdS. The 't Hooft limit involves taking the asymptotic string coupling $\tilde{g}_s = \tilde{g}^2/4\pi$ to zero. For NS5 branes embedded in asymptotically flat space a similar limit decouples the degrees of freedom on the fivebranes. The resulting theory is precisely Little String Theory. We will show that the 't Hooft limit has the same effect in the present case. The effective string coupling goes to zero everywhere except in a region very close to the fivebranes. In this region the solution coincides with the near horizon geometry of m toroidally wrapped NS fivebranes with no other SUGRA fields turned on. This geometry is holographically dual to LST [25].

The decoupling described above has a simple interpretation in the dual confining gauge theory. The Hilbert space of the large-N theory contains a sector where states form towers of Kaluza-Klein modes. These states and their interactions respect six-dimensional Lorentz invariance. The theory also contain another sector of states which badly violate the six-dimensional Lorentz invariance. At large but finite n the two sectors are weakly-coupled to each other. In the 't Hooft large-N limit, the states in the four-dimensional sector retain finite masses but decouple completely both from each other and from the six-dimensional sector. The six-dimensional sector remains interacting and is exactly Little String Theory. Our main result can therefore be summarised as follows:

Result. We consider the β-deformation of $\mathcal{N}=4$ SUSY Yang-Mills with gauge group $\mathrm{U}(mn)$, gauge coupling $\tilde{g}^2$ and deformation parameter $\tilde{\beta}=8\pi^2 i/\tilde{g}^2 n$. The theory has a vacuum where the $\mathrm{U}(mn)$ gauge group is confined down to a $\mathrm{U}(m)$ subgroup at scale $\tilde{\mathrm{v}}$. In the limit $n\to\infty$, $\tilde{g}^2\to 0$ with $\tilde{g}^2 n$, m and $\tilde{\mathrm{v}}$ fixed, the interacting sector of the theory is equivalent to Little String Theory with low energy gauge coupling $G_6\sim\sqrt{\tilde{g}^2 n}\,\tilde{\mathrm{v}}^{-1}$ compactified on a torus of radii $R\sim\tilde{g}^2 n\tilde{\mathrm{v}}^{-1}$.

The precise statement of this result is given in section 8 below. The supergravity analysis which establishes this result is valid for $\tilde{g}^2 n\gg m$ where the compactification torus is large compared to the gauge coupling of LST. We will also conjecture that the result holds for more general values of $\tilde{g}^2 n$ although the evidence for this is limited.

The results described in this paper reveal a new aspect of the duality between large-N confining gauge theory and string theory. As with other dualities, it usefulness depends on identifying regions in parameter space where calculations can be performed on (at least) one side of the correspondence. On the string theory side, the theory on m NS5 branes has an interesting double-scaling limit [26, 27]. The resulting so-called Double-Scaled Little String Theory is holographically dual to a weakly-coupled IIB background with an exactly solvable worldsheet conformal field theory. In section 7, we adapt the results described above to deconstruct DSLST. The dual gauge theory is realised in a phase where $\mathrm{U}(mn)$ is confined down to $\mathrm{U}(1)^m$. The weak coupling regime of DSLST corresponds to a regime where all states charged under the low-energy gauge symmetry are very massive. We argue that the standard large-N scaling arguments for confining gauge theories apply in this regime and should lead to an infinite tower of weakly interacting glueball states with an S-matrix exhibiting Regge behaviour. We find that DSLST provides, for the first time, an analytic description of this phenomenon in terms of weakly-coupled closed string theory.

Another aspect of the duality described above is that it provides a new non-perturbative definition of LST and therefore, via holography, of linear dilaton backgrounds. Although we will not develop this viewpoint much in the present paper, we make some preliminary comments in the final section and hope to return to this topic in future work. The rest of the paper is organised as follows. In section 2, we review some of the properties of LST and also introduce a more general class of fivebrane theories which reduce to non-commutative gauge theories at low energy. Section 3 reviews the β-deformation of $\mathcal{N}=4$ SUSY Yang-Mills including some of the main results of [9]. In section 4 we review the basic idea of deconstruction at the classical level. In section 5 we take a first look at the continuum limit. Section 6 is devoted to constructing a string theory dual of the Higgs branch vacuum and section 7 reviews the corresponding dual for the confining phase. In section 8 we discuss the continuum limit and formulate a precise version of the deconstruction conjecture. In section 9 we adapt our results to the case of DSLST described above. Some calculational details from sections 6 and 9 are relegated to appendices A and B respectively.

JHEP07(2004)016

2. LST and its non-commutative cousins

The basic definition of Little String Theory (LST) is as a decoupling limit of the world-volume theory of fivebranes in type-II string theory. We will start from m parallel D5 branes of the type-IIB theory with string coupling g_s and squared string length α'. At low-energy the theory on the world volume reduces to a six dimensional U(m) gauge theory with $\mathcal{N} = (1,1)$ supersymmetry and gauge coupling $\hat{G}_6 = \sqrt{16\pi^3\alpha' g_s}$. As for the lower dimensional Dirichlet branes we can try to take a limit which isolates the worldvolume theory. To decouple the excited modes of the open strings ending on the D5's we need to take the limit $\alpha' \to 0$. If we also try to keep the low-energy gauge coupling fixed we are forced to simultaneously take the limit $g_s \to \infty$.

We can understand this limit better using the S-duality transformation of the IIB theory which acts on the parameters as,

$$g_s \to \tilde{g}_s = \frac{1}{g_s} \qquad \alpha' \to \tilde{\alpha}' = g_s\alpha' \, . \tag{2.1}$$

This transformation maps the m D5 branes we started with to a configuration of m parallel NS5 branes. In terms of the S-dual variables, the low energy gauge coupling is $\hat{G}_6 = \sqrt{16\pi^3\tilde{\alpha}'}$ and the decoupling limit becomes simply $\tilde{g}_s \to 0$ with $\tilde{\alpha}'$ held fixed. In this limit the ten-dimensional Planck length goes to zero and the theory on the branes is decoupled from gravity. We will now briefly review the basic properties of the resulting theory:

1. The IIB LST has six-dimensional $\mathcal{N} = (1,1)$ super-Poincare invariance. The theory also has an exact SO(4) R-symmetry corresponding to rotations of the four transverse directions of the branes.

2. Apart from the integer m, the theory has a single parameter, a characteristic mass scale which in our conventions is $\hat{M}_s = 1/\sqrt{16\pi^3\tilde{\alpha}'}$. At energies below the scale $\hat{M}_s/\sqrt{m}$, it reduces to $\mathcal{N} = (1,1)$ supersymmetric Yang-Mills theory in six dimensions with gauge coupling $\hat{G}_6 = 1/\hat{M}_s$. The resulting low-energy gauge group is U(m). The U(1) vector multiplet corresponding to the center of U(m) is completely decoupled and the remaining interacting sector of the theory has low-energy gauge-group SU(m)

3. The theory has a moduli space $\mathrm{Sym}^m R^4$ corresponding to the Coulomb branch of the low-energy gauge theory. Away from the origin the U(m) gauge symmetry is broken to $\mathrm{U}(1)^m$.

4. In addition to the massless gauge multiplet, the spectrum of the U(m) theory contains BPS saturated strings with tension $\hat{T} = 8\pi^2\hat{M}_s^2$. Roughly speaking these strings correspond to bound states of the IIB string with the NS5 brane.

5. When the theory is compactified on a circle the existence of string winding modes leads to an exact T-duality relating IIB LST and the corresponding LST on the IIA NS fivebrane. This property indicates that LST is not a local quantum field theory.

JHEP07(2004)016

It will be useful to introduce a slightly more general class of six-dimensional theories first considered in [28]. The theories in question reduce to *non-commutative* gauge theories with $\mathcal{N} = (1,1)$ SUSY in the IR. They are obtained by taking an appropriate decoupling limit on the worldvolume of type-IIB D5 branes with a non-zero background for $B_{\rm NS}$. These theories played an important role in the analysis of the $\mathcal{N} = 1^*$ theory in [15] . They will also enter in our analysis of deconstruction although we emphasize that our final results concern conventional commutative LST.

We start from a configuration consisting of m flat D5 branes extended in the 0, 1, 2, 3, 4, 5 directions of $R^{9,1}$. In addition we will introduce a constant two-form potential in the y_4-y_5 plane,

$$B_{\rm NS} = \tan\varphi\, dy_4 \wedge dy_5\,. \tag{2.2}$$

The supergravity solution for this configuration was given in [28]. The string frame metric, RR four-form, NS two-form, and dilaton fields read,

$$\begin{aligned}
ds^2 &= f^{-\frac{1}{2}}(u)\left[\eta_{\mu\nu}dy^\mu dy^\nu + h(u)\left(dy_4^2 + dy_5^2\right)\right] + \alpha'^2 f^{\frac{1}{2}}(u)\left[du^2 + u^2 d\Omega_3^2\right]\\
B_{\rm NS} &= \tan\varphi f^{-1}(u)h(u)\, dy_4 \wedge dy_5\\
\exp(2\phi) &= g_s^2 f^{-1}(u)h(u)\\
\chi_4 &= \frac{1}{g_s}\sin\varphi f^{-1}(u)\, dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3
\end{aligned} \tag{2.3}$$

where $u = s/\alpha'$ is a rescaling of the distance, $s = \sqrt{y_6^2 + \cdots + y_9^2}$, from the branes in the four transverse directions. The functions of the radial variable u appearing in the solution are given by,

$$\begin{aligned}
f(u) &= 1 + \frac{R^2}{\alpha'^2 u^2}\\
h^{-1}(u) &= \sin^2\varphi\, f^{-1}(u) + \cos^2\varphi
\end{aligned} \tag{2.4}$$

with $R^2 = g_s\alpha' m/\cos\varphi$. The self-duality of the RR five-form field strength is imposed by setting $F_5 = d\chi_4 + \star d\chi_4$. There are also m units of RR three-form flux $F_{(3)}$ through a three-sphere surrounding the branes but the explicit form of this field will not be needed in the following.

We will now take a decoupling limit of the sort introduced by Seiberg and Witten [29]. Specifically we take the $\alpha' \to 0$ limit with g_s and $b = \alpha' \tan\varphi$ held fixed. The SW limit also requires us to rescale the coordinates y_5 and y_6 in the directions of non-zero $B_{\rm NS}$ field. Specifically we define $\bar{y}_4 = (b/\alpha')y_4$ and $\bar{y}_5 = (b/\alpha')y_5$ and hold $\bar{y}_4$, $\bar{y}_5$ and the rescaled radial coordinate u fixed as $\alpha' \to 0$. Standard arguments based on the quantisation of open strings ending on the $D5$ brane tell us that the low-energy theory on the brane is maximally supersymmetric Yang-Mills theory in $5+1$ dimensions with gauge coupling $G_6^2 = 16\pi^3 g_s b$. The theory has non-commutativity in the $\bar{y}_4$-$\bar{y}_5$ plane: $[\bar{y}_4, \bar{y}_5] = 2\pi i\, b$.

In the limit of interest, the excited modes of the open string decouple as α' goes to zero. As the ten-dimensional Planck mass goes to infinity in this limit we also expect the theory to decouple from gravity.[4] Thus the final result is a non-gravitational theory which reduces to six-dimensional non-commutative U(m) gauge theory at low energies. For brevity we will denote the resulting decoupled world-volume theory as $\mathcal{T}[M_s, g_s]$. This theory has a characteristic mass scale $M_s = 1/\sqrt{16\pi^3 g_s b} = 1/G_6$. In addition to the elementary excitations of the gauge fields, the theory also contains BPS strings of tension $T = 8\pi^2 M_s^2$ corresponding to non-commutative Yang-Mills instantons embedded in six dimensions. In string theory language, these correspond to D-strings bound to the D5 branes.

Note that string coupling g_s is held fixed in the decoupling limit and remains as an additional parameter of the theory. This is different from the case of D5 branes without background $B_{\rm NS}$ considered above where the only possible decoupling limit involves taking the limit $g_s \to \infty$ leading to the conventional definition of LST. The energy scale above which the non-commutativity in the $\bar{y}_4$-$\bar{y}_5$ plane becomes important is[5] $M_{\rm NC} = 1/\sqrt{2\pi b} = \sqrt{8\pi^2 g_s} M_s$. The role of the extra parameter g_s is therefore to set the ratio $M_{\rm NC}/M_s$. This suggests that non-commutativity should disappear in the strong coupling limit $g_s \to \infty$. We will study this limit more carefully below and see that it yields ordinary (ie commutative) LST.

We will now consider different regimes in which the decoupled fivebrane theory has a weakly coupled effective description. Like any non-abelian gauge theory in six dimensions the low-energy gauge theory description of $\mathcal{T}[M_s, g_s]$ becomes strongly coupled in the UV at scales above that set by the inverse 't Hooft coupling $M_s/\sqrt{m}$ and perturbation theory breaks down. To understand the behaviour of the theory in this regime we can consider instead the dual gravitational background. In the decoupling limit discussed above the D5 solution becomes,

$$\begin{aligned} ds^2 &= \alpha' \left(\frac{au}{b}\right) \left[\eta_{\mu\nu} dy^\mu dy^\nu + \bar{h}(u) \left(d\bar{y}_4^2 + d\bar{y}_5^2\right)\right] + \alpha' \left(\frac{b}{au}\right) \left[du^2 + u^2 d\Omega_3^2\right] \\ B_{\rm NS} &= \frac{\alpha'}{b}\, a^2 u^2 \bar{h}(u)\; d\bar{y}_4 \wedge d\bar{y}_5 \\ \exp(2\phi) &= g_s^2 a^2 u^2 \bar{h}(u) \\ \chi_4 &= \frac{1}{g_s} \left(\frac{\alpha'^2 a^2 u^2}{b^2}\right) dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \end{aligned} \tag{2.5}$$

where,

$$\bar{h}(u) = \frac{1}{1 + a^2 u^2} \qquad \text{with } a^2 = \frac{b}{g_s m}\,. \tag{2.6}$$

As before there are also m units of RR threeform flux through an S^3 surrounding the D5 branes which we have not shown explicitly.

The rescaled radial coordinate $u = s/\alpha'$ corresponds to the energy of streched fundamental strings ending on a probe D5 brane placed at fixed radial position s. By analogy

[4] The issue of whether gravity really decouples in such a limit is often subtle. Whether or not this is the case for the family of five-brane worldvolume theories considered in this section will not affect the main conclusions of the paper.

[5] More precisely this formula for $M_{\rm NC}$ applies only when the corresponding six-dimensional gauge theory is weakly coupled. This is the case provided $M_{\rm NC} \ll M_s/\sqrt{m}$.

with the UV/IR correspondence of more familiar conformal examples of holography, it is tempting to interpret dependence on the coordinate u as RG flow in the worldvolume theory. As emphasised in [24], there is no direct generalization of the UV/IR correspondence for the near horizon geometry fivebranes. The energy scale corresponding to a fixed value of u depends on the process considered.

We can now determine when the dual supergravity background (2.5) is weakly coupled. The validity of tree-level string theory in the above background requires that the effective string coupling e^{ϕ} is small everywhere. The dilaton solution in (2.5) shows that e^{ϕ} is a monotonically increasing function of the radial coordinate u which tends to the constant value g_s. Hence provided $g_s \ll 1$ the dilaton is small everywhere and string loops are supressed. The supergravity approximation to string theory is valid as long as the curvature of the solution is small. This is the case provided that,

$$u \gg \frac{1}{\sqrt{g_s b m}} = \frac{M_s}{\sqrt{m}} \,. \tag{2.7}$$

For smaller values of u the curvature becomes large and the theory is better described by weakly-coupled six-dimensional non-commutative Yang-Mills theory discussed above. With a naive interpretation of u as an energy scale in the worldvolume theory, this matches the fact that the low-energy gauge theory becomes strongly-coupled in the UV at the scale $M_s/\sqrt{m}$ set by the inverse 't Hooft coupling. In other words the domains of validity of supergravity and Yang-Mills theory are exactly complimentary.

As the D5 brane configuration we started from is BPS saturated and exists for all values of the string coupling we will assume that the world volume theory $\mathcal{T}[g_s, M_s]$ also makes sense for all values of g_s. Another regime where we can hope to study the theory succesfully is that of very large g_s which we can map to weak coupling via the S-duality of the IIB theory. Specifically the fields and parameters transform as;

$$\begin{aligned} g_s \to \tilde{g}_s &= \frac{1}{g_s} & \qquad \alpha' \to \tilde{\alpha}' &= g_s \alpha' \\ \exp(\phi) \to \exp(\tilde{\phi}) &= \exp(-\phi) & \qquad ds^2 \to d\tilde{s}^2 &= g_s \exp(-\phi) ds^2 \,. \end{aligned} \tag{2.8}$$

Under these transformations the background (2.5) gets mapped to,

$$\begin{aligned} d\tilde{s}^2 &= \left(\frac{\tilde{\alpha}'\tilde{g}_s}{b}\right) \bar{h}^{-\frac{1}{2}}(u) \left[\eta_{\mu\nu} dy^\mu dy^\nu + \bar{h}(u) \left(d\bar{y}_4^2 + d\bar{y}_5^2\right) + \left(\frac{b^2}{a^2u^2}\right)\left(du^2 + u^2 d\Omega_3^2\right)\right] \\ B_{\rm RR} &= \tilde{g}_s \tilde{\alpha}' \left(\frac{a^2u^2}{b}\right) \bar{h}(u)\, d\bar{y}_4 \wedge d\bar{y}_5 \\ \exp(2\tilde{\phi}) &= \left(\frac{\tilde{g}_s^2}{a^2u^2}\right) \bar{h}^{-1}(u) \\ \chi_4 &= \frac{1}{\tilde{g}_s} \left(\frac{\tilde{\alpha}'^2 \tilde{g}^2 a^2 u^2}{b^2}\right) dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \,. \end{aligned} \tag{2.9}$$

The solution also has m units of flux for the NS three-form field stength through an S^3 surrounding the branes. In terms of the new variables, the radial coordinate is $u = r/\tilde{g}_s \tilde{\alpha}'$

JHEP07(2004)016

which corresponds to the energy of stretched D-strings ending on a probe NS5 brane placed a distance r from the other branes. The functions of u appearing in (2.9) are,

$$\bar{h}(u) = \frac{1}{1+a^2u^2} \qquad \text{with } a^2 = \frac{b\tilde{g}_s}{m} \tag{2.10}$$

As before we can determine the regime where the dual supergravity background is weakly coupled. The validity of tree-level string theory requires that the effective string coupling $e^{\tilde{\phi}}$ is small. In the background (2.9) the dilaton is a monotonically decreasing function of the radial coordinate u which tends to a constant value $e^{\tilde{\phi}_\infty} = \tilde{g}_s$ at $u = \infty$. Hence for $\tilde{g}_s \ll 1$ tree-level string theory is valid for large-u which (roughly) corresponds to the UV region of the the theory on the brane. The dilaton becomes order one at a scale,

$$u \sim \sqrt{\frac{m\tilde{g}_s}{b}} = \sqrt{m}M_s\,. \tag{2.11}$$

For values of u below this scale we should undo the S-duality transformation and return to the D5-brane background. The supergravity approximation is valid as long as the curvature of the solution is small. This is the case provided that,

$$u \gg \sqrt{\frac{\tilde{g}_s}{bm}} = \frac{M_s}{\sqrt{m}} \tag{2.12}$$

We find that the SUGRA solution undergoes an important transition at the scale,

$$u \sim a^{-1} = \frac{\sqrt{m}}{\tilde{g}_s}M_s\,. \tag{2.13}$$

Notice that for $\tilde{g}_s \ll 1$ this scale is higher than the scales (2.11), (2.12) indicating that the crossover scale lies inside the regime of validity of classical supergravity. When $u \ll a^{-1}$ we have $h(u) \simeq 1$ and the familiar throat region of the NS5 brane solution appears. In this region, the solution can be written as,

$$\begin{aligned} d\tilde{s}^2 &= \eta_{AB}dY^AdY^B + \tilde{\alpha}' m\frac{d\hat{u}^2}{\hat{u}^2} + \tilde{\alpha}' m d\Omega_3^2 \\ \exp(2\tilde{\phi}) &= \frac{m}{\tilde{\alpha}'}\frac{1}{\hat{u}^2} \end{aligned} \tag{2.14}$$

where η_{AB} with $A, B = 0, 1, \ldots, 5$ is the standard flat metric on six-dimensional Minkowski space. Note that we have defined rescaled coordinates along the brane as $Y_A = (M_s/\hat{M}_s)y_\mu$ for $A = \mu = 0, 1, 2, 3$, and $Y_A = (M_s/\hat{M}_s)\bar{y}_{4,5}$ for $A = 4, 5$. We have also rescaled the radial coordinate as $\hat{u} = (\hat{M}_s/M_s)u$ where $\hat{M}_s = 1/\sqrt{16\pi^3\tilde{\alpha}'}$.

As before we have m units of NS 3-form flux through an S^3 surrounding the branes. However, all the other fields of IIB supergravity go to zero rapidly for $u \ll a^{-1}$. Importantly, the solution in this region therefore has no non-zero RR field strengths. In fact the solution (2.14), is simply the near horizon geometry of m NS5 branes with no additional fields turned on. The geometry in question is known to be holographically dual to ordinary IIB Little String Theory with low-energy gauge group SU(m) [25]. As above, the dynamics of the latter theory is characterised by the mass scale $\hat{M}_s = 1/\sqrt{16\pi^3\tilde{\alpha}'}$. The rescaling of the coordinates described mean that lengths and energies measured in LST are not the same as those of the theory $\mathcal{T}[g_s, M_s]$ but differ by appropriate powers of $M_s/\hat{M}_s$. The net effect of this rescaling is to replace the mass parameter $\hat{M}_s$ of LST with the mass parameter M_s of the theory worldvolume theory $\mathcal{T}[g_s, M_s]$.

Thus we find that, when $g_s \gg 1$, the holographic description of our non-commutative world-volume theory $\mathcal{T}[M_s, g_s]$ simplifies for small values of the radial coordinate u. States localised far down the throat at $u \ll a^{-1}$ are effectively describe by LST with low energy gauge group SU(m) and mass parameter M_s. Roughly speaking this is a strong coupling analog of the fact that non-commutative gauge theory reduces to its commutative counter part at low energies. However the analogy is imprecise because, as mentioned above, the relation between the radial coordinate and energy in the boundary theory is ambiguous.

It will be useful to understand what happens if we now take the limit $g_s \to \infty$ with the scale M_s fixed. As the S-dual coupling $\tilde{g}_s$ goes to zero the string coupling vanishes every where except for values of the radial coordinate u satisfying $u \ll a^{-1}$ for which the geometry is given by the NS5 brane throat solution (2.14). Thus the states located in the throat region retain finite interactions while all other states decouple. The upshot is that the resulting theory $\mathcal{T}[M_s, g_s \to \infty]$ includes the IIB LST with low-energy gauge group SU(m) along with other states which are completely decoupled. Among the decoupled states are the U(1) gauge boson corresponding to the center of the original gauge group U(m) and its $\mathcal{N} = (1, 1)$ superpartners.

3. Review of the β-deformation

In this section we review the key features of the β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills theory studied in [9]. In terms of $\mathcal{N} = 1$ superfields, this theory contains a U(N) vector multiplet V and three chiral multiplets Φ_i, with $i = 1, 2, 3$, in the adjoint representation of the gauge group. The classical superpotential is given as,

$$\mathcal{W} = i\kappa \, \mathrm{Tr}_N \left(\Phi_1[\Phi_2, \Phi_3]_\beta\right) \tag{3.1}$$

where,

$$[\Phi_i, \Phi_j]_\beta = \exp\left(i\frac{\beta}{2}\right) \Phi_i\Phi_j - \exp\left(-i\frac{\beta}{2}\right) \Phi_j\Phi_i \, . \tag{3.2}$$

The $\mathcal{N} = 4$ theory is recovered for $\beta = 0$ and $\kappa = 1$. Apart from the complex parameters β and κ appearing in the superpotential, the theory also depends on the complexified gauge coupling $\tau = 4\pi i/g^2 + \vartheta/2\pi$. We will now review the classical and quantum properties of this theory in turn.

3.1 The classical theory

In the classical theory the complex parameter κ has no effect and it can be set to one. In contrast the classical vacuum structure of the theory depends strongly on the deformation parameter β. In particular, new Higgs branches appear at special values of β. We will focus on one of these branches which occurs when $\beta = 2\pi/n$ where n is a divisor of $N = mn$. In this case the theory has a Higgs branch (denoted $\mathcal{H}_m$ in [9]) where the gauge symmetry is broken down to a $\mathrm{U}(m)$ subgroup. The scalar expectation values on this branch are given as,

$$\langle\Phi_1\rangle = \alpha_1 I_{(m)} \otimes U_{(n)} \qquad \langle\Phi_2\rangle = \alpha_2 I_{(m)} \otimes V_{(n)} \qquad \langle\Phi_3\rangle = \alpha_3 I_{(m)} \otimes V^\dagger_{(n)} U^\dagger_{(n)} \tag{3.3}$$

where α_1, α_2 and α_3 are complex numbers. Here $I_{(m)}$ denotes the $m \times m$ unit matrix, and $U_{(n)}$ and $V_{(n)}$ are the $n \times n$ clock and shift matrices which are given explicitly as,

$$\left(U_{(n)}\right)_{ab} = \delta_{a,b}\omega^a_{(n)} \qquad \left(V_{(n)}\right)_{ab} = \delta^{(n)}_{a,b-1} \tag{3.4}$$

where $\omega_{(n)} = \exp(2\pi i/n)$ is an n'th root of unity, and $\delta^{(n)}$ denotes a modified Kronecker δ which is one if its two indices are equal modulo n and is zero otherwise. An alternative way of specifying a vacuum state on $\mathcal{H}_m$ is by giving the expectation values of three independent gauge-invariant chiral operators,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_1^n\right]\right\rangle = \alpha_1^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_2^n\right]\right\rangle = \alpha_2^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_3^n\right]\right\rangle = \alpha_3^n\,. \tag{3.5}$$

The full Higgs branch $\mathcal{H}_m$ is the three dimensional complex orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. The Higgs branch can be realized in string theory by placing m coincident D3 branes at a point in the transverse space of a $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ singularity with one unit of discrete torsion. The effective action for the massless degrees of freedom on this branch is $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\mathrm{U}(m)$ and complex gauge coupling $n\tau$. We can also find vacua where the unbroken gauge symmetry is $\mathrm{U}(1)^m$ by moving onto the Coulomb branch of the low-energy theory. This corresponds to separating the m D3 branes in the string theory set-up.

One way to characterise the different phases of a gauge theory is to probe the theory with external electric and magnetic charges. As usual, the possible electric charges are classified by the center of the gauge group [30]. The center of $\mathrm{U}(mn)$ is $\mathrm{U}(1)$ and the corresponding electric charge is denoted $q \in \mathbb{Z}$. Possible magnetic charges are classified by first homotopy class of the gauge group: $\pi_1(\mathrm{U}(mn)) \simeq \mathbb{Z}$. We denote the corresponding magnetic charge $\tilde{q} \in \mathbb{Z}$. If we move onto the Higgs branch, the vacuum condensate leads to the screening of external electric charges. As a $\mathrm{U}(m)$ subgroup remains unbroken, only electric charges with $q = 0 \mod m$ are competely screened. Charges $q \neq 0 \mod n$ will produce long-range Coulomb fields. The Higgs mechanism also leads to the confinement of external magnetic charges by the formation of chromomagnetic flux tubes. The unbroken $\mathrm{U}(m)$ subgroup means that confinement is only partial. In particular those magnetic charges with $\tilde{q} = 0 \mod n$, produce long-range magnetic fields and remain unconfined.

3.2 The quantum theory

The quantum theory corresponding to the β-deformed superpotential (3.1) has several remarkable properties. Firstly it corresponds to an exactly marginal deformation of $\mathcal{N} = 4$ SUSY Yang-Mills. More precisely, the theory has a critical surface in coupling constant space defined by $\kappa = \kappa_{cr}[\tau, \beta]$ on which the all β-functions vanish and conformal invariance is unbroken. The critical surface includes the $\mathcal{N} = 4$ line parametrized by the gauge coupling τ with $\beta = 0$, $\kappa = 1$. Thus we have a two-complex parameter family of $\mathcal{N} = 1$ superconformal theories.

JHEP07(2004)016

As usual for $\mathcal{N}=1$ theories the exact vacuum structure is determined by the F-terms in the effective action which are holomorphic in the complex parameters β and τ. An exact solution for the holomorphic sector of the theory was obtained in [19] and applied to the Higgs branch theory in [9]. One result is that the classical Higgs branch discussed in the previous section persists in the quantum theory for all values of the gauge coupling. The exact low energy gauge coupling on this branch is not renormalised and takes its classical value $n\tau$.

As reviewed in [9], the exact SL$(2,\mathbb{Z})$ duality of the $\mathcal{N}=4$ theory extends to the β-deformed theory. To be precise, the duality acts on a renormalized gauge coupling;

$$\tau_R = \frac{4\pi i}{g_R^2} + \frac{\vartheta_R}{2\pi} = \tau + \frac{iN}{\pi}\log\kappa \tag{3.6}$$

and also on the deformation parameters as

$$\tau_R \to \frac{a\tau_R + b}{c\tau_R + d} \qquad \beta \to \frac{\beta}{c\tau_R + d} \qquad \kappa^2 \sin\beta \to \frac{\kappa^2\sin\beta}{c\tau_R + d}\,. \tag{3.7}$$

The algebraic renormalization of the coupling given in (3.6) plays no role in the following and we will ignore it from now on and supress the subscript on τ_R.

As usual the S-generator of SL$(2,\mathbb{Z})$, which acts as $\tau \to -1/\tau$ also interchanges electric and magnetic charges. Under this transformation, the Higgs phase vacuum where the gauge group is spontaneously broken down to a U(m) subgroup is mapped to a confining phase with an unconfined U(m) subgroup. Again we can characterise this phase by probing it with external charges. The S-duality transformation interchanges the integers q and $\tilde{q}$ which characterize the possible electric and magnetic charges respectively. The magnetic condensate in the confining vacuum means that external magntic charges with $\tilde{q}=0$ mod m are completely screened. Conversely, external electric charges are confined by the formation of chromoelectric flux tubes unless $q=0$ mod n. This phase is unusual because as it exhibits both electric confinement and spontaneously broken conformal invariance. Note that Higgs and Confining phases are genuinely different as expected in a theory containing only adjoint fields.

Let us consider the action of the S-generator of SL$(2,\mathbb{Z})$ in the case $\vartheta=0$. The transformation relates the theory with parameters g^2 and $\beta=2\pi/n$ and chiral fields Φ_i to a dual theory with corresponding parameters $\tilde{g}^2=16\pi^2/g^2$ and $\tilde{\beta}=8\pi^2 i/\tilde{g}^2 n$ and chiral fields $\tilde{\Phi}_i$. The former theory has a Higgs branch. If we consider the Higgs branch vacuum with VEVs for Φ_i as given in (3.3), (3.5) above, then the S-dual vacuum has non-vanishing chiral VEVs,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_1^n\right]\right\rangle = \tilde{\alpha}_1^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_2^n\right]\right\rangle = \tilde{\alpha}_2^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_3^n\right]\right\rangle = \tilde{\alpha}_3^n\,. \tag{3.8}$$

The non-trivial modular weights of the chiral operators [20] appearing in (3.8) imply that $\tilde{\alpha}_i=(\tilde{g}^2/4\pi)\alpha_i$ for $i=1,2,3$ [9].

As explained in [9], the scalar expectation values in (3.8) do not correspond to any vacuum of the classical theory. At first sight the existence of this vacuum seems to lead to a contradiction for $\tilde{g}^2 n \ll 1$ where one might expect the classical analysis to be valid. However, in this regime, the theory is weakly coupled only in the sense that the gauge coupling is small. In contrast, the deformation parameter $\tilde{\beta}$ is large and imaginary so that the lagrangian includes exponentially large Yukawa couplings as well as quartic self-couplings of the adjoint scalars. Thus quantum corrections involving the adjoint scalars and their $\mathcal{N} = 1$ superpartners are not supressed and classical analysis is invalid. The confining phase vacua therefore lie on a quantum branch which is invisible classically.

4. Classical deconstruction

In [9], the classical spectrum and effective action of the theory on the Higgs branch $\mathcal{H}_m$ was determined. As in [9], we will consider the vacuum state on $\mathcal{H}_m$ specified by VEVs (3.5) and, for simplicity, set $\alpha_3 = 0$. In this vacuum, the exact classical mass formula for each adjoint field is,

$$M^2 = 4|\alpha_1|^2 \sin^2\left(\frac{l_1\pi}{n}\right) + 4|\alpha_2|^2 \sin^2\left(\frac{l_2\pi}{n}\right) \tag{4.1}$$

with integers l_1, $l_2 = 1, 2, \ldots, n$.

Deconstruction starts from the observation that, for large-n, (4.1) coincides with the spectrum of KK modes of a six-dimensional theory compactified to four dimensions on a torus. The integers l_1 and l_2 correspond to the quantized momenta around the two compact directions. At finite n, (4.1) matches the truncated KK tower we would find if the extra dimensions were discretized on an $n \times n$ lattice. The appearance of extra dimensions is not limited to the spectrum but also extends to the classical action which can actually be rewritten as a six-dimensional gauge theory. In fact, the classical theory at fixed n can be understood as a non-commutative $\mathrm{U}(m)$ lattice gauge theory [31, 32] defined on $R^{3,1} \times \mathcal{L}$, where $\mathcal{L}$ is an $n \times n$ lattice with periodic boundary conditions for all the fields. The lattice theory in question is a discretization of $\mathcal{N} = (1,1)$ supersymmetric non-commutative Yang-Mills theory with gauge group $\mathrm{U}(m)$ compactified down to four dimensions on a torus.

The parameters of the six dimensional theory can be expressed in terms of the four-dimensional parameters as follows. The lattice spacings of the two compact discrete dimensions are,

$$\varepsilon_1 = \frac{1}{|\alpha_1|} \qquad \varepsilon_2 = \frac{1}{|\alpha_2|} \tag{4.2}$$

The radii of the two-dimensional torus which they define are,

$$R_1 = \frac{n}{2\pi|\alpha_1|} \qquad R_2 = \frac{n}{2\pi|\alpha_2|} \tag{4.3}$$

The six-dimensional gauge coupling and non-commutivity parameter are given by,

$$G_6^2 = \frac{g^2 n}{|\alpha_1||\alpha_2|} \qquad \theta = \frac{n}{2\pi|\alpha_1||\alpha_2|} \tag{4.4}$$

respectively.

JHEP07(2004)016

The derivation of these relations given in [9] was purely classical and, a priori they are only reliable at weak coupling $g^2N \ll 1$. The classical theory is characterised by the following heierarchy of length scales: $G_6 \ll \varepsilon_i \ll \sqrt{\theta} \ll R_i$. Provided we consider $n \gg 1$, these scales are well seperated and it makes sense to write down a six-dimensional continuum effective valid on length-scales much larger than the lattice spacing. This continuum effective action is precisely six-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory with gauge group $\mathrm{U}(m)$ defined on $R^{3,1} \times T^2_\Theta$. Here T^2_Θ is the non-commutative torus with dimensionless non-commutativity parameter $\Theta = 1/n$. The fact that $G_6 \ll \varepsilon_i$ shows that this effective continuum gauge theory is weakly coupled throughout its range of validity.

A striking feature of the low-energy effective theory described above is that it has sixteen supercharges. In contrast the microscopic four-dimensional theory we started with only has four supercharges. The classical spectrum of W-bosons are BPS saturated with respect to the enlarged supersymmetry of the low-energy theory. The low-energy theory described above also has BPS saturated soliton strings which were studied in detail in [9]. These correspond to $\mathrm{U}(m)$ Yang-Mills instantons on $R^2 \times T^2_\Theta$ [35] embedded as static field configurations in six dimensions. The lightest strings, corresponding to instantons of topological charge one, have tension,

$$T = \frac{8\pi^2}{G_6^2} = \frac{8\pi^2|\alpha_1||\alpha_2|}{g^2 n}\,. \tag{4.5}$$

In terms of the underlying four-dimensional $\mathrm{U}(N)$ gauge theory these strings are precisely the expected chromomagnetic flux tubes which confine external magnetic charges. In particular, an instanton string of topological charge k can end on and external magntic charge $\tilde{q} = k \bmod n$.

5. A first look at the continuum limit

Given any lattice theory it is natural to question whether we can find an interacting continuum limit. In the present context this means a limit in which the lattice spacing goes to zero, while the six-dimensional gauge coupling is held fixed. One interesting limit discussed in [9] is,[6]

Limit I: we consider the $\mathrm{U}(mn)$ theory with $\beta = 2\pi/n$ in the Higgs branch vacuum specified in (3.5) above and take the limit $n \to \infty$, $g^2 \to \infty$ while holding m and g^2/n fixed We also scale the VEVs as

$$|\alpha_1| \sim n \to \infty \qquad |\alpha_2| \sim n \to \infty \tag{5.1}$$

Using the results (4.2), (4.3), (4.4) for the six-dimensional parameters, we find that the lattice spacings ε_i and the non-commutativity parameter θ go to zero, while the six-dimensional gauge coupling G_6 and the radii of compactification R_i remain fixed. Naively this indicates that we end up with a commutative continuum theory defined on $R^{3,1} \times T^2$.

[6] This is the limit discussed in section 9.2 of [9]. A different continuum limit which yields a non-commutative theory was also discussed in [9] but we will not consider it here.

Of course Limit I, as defined above, is a strong coupling limit and we should immediately question whether it is legitimate to extrapolate the classical formulae (4.2), (4.3), (4.4) which were derived assuming weak coupling. We will postpone this discussion momentarily and take Limit I at face value as a candidate continuum limit.

The next question is what continuum theory could possibly arise in this limit. The naive answer based purely on our classical analysis is that we find a conventional $\mathrm{U}(m)$ gauge theory with $\mathcal{N} = (1,1)$ supersymmetry in six dimensions. This cannot be the whole story as such a theory is certainly non-renormalisable and requires a consistent UV completion to make sense. In fact there is only one candidate theory which provides such a completion without also coupling the low-energy theory to gravity. This is the Little String Theory discussed in section 2 above. The low-energy gauge group of LST is $\mathrm{SU}(m)$ rather than $\mathrm{U}(m)$ so we also need to include an additional free $\mathrm{U}(1)$ vector multiplet of $\mathcal{N} = (1,1)$ SUSY. The simplest deconstruction conjecture is therefore,

Conjecture: when we take Limit I, the theory on the Higgs branch with $\beta = 2\pi/n$ becomes type-IIB LST on $R^{3,1} \times T^2$ plus an additional decoupled $\mathrm{U}(1)$ vector multiplet. The mass scale of LST is given by,

$$M_s = \sqrt{\frac{|\alpha_1||\alpha_2|}{g^2 n}} \tag{5.2}$$

which remains fixed in Limit I, as do the radii of compactification R_1 and R_2 given in (4.3) above.

At this stage, the motivation for the conjecture depends on our extrapolation of the classical formulae (4.2), (4.3), (4.4) to strong coupling. As the classical low-energy effective action has sixteen supercharges it is tempting to try to invoke non-renormalisation theorems to justify this extrapolation. In particular, the spectrum of W-bosons which represent the Kaluza-Klein modes of the six-dimensional effective theory are BPS saturated with respect to this enlarged supersymmetry. The mass spectrum of these states dictates both the radii of compactification and the lattice spacing. As the masses of BPS states are protected from quantum corrections in a theory with sixteen supercharges, we might hope to infer that the classical formulae (4.2), (4.3) are exact.

The argument given above is far from convincing. Although the low-energy effective action has sixteen supercharges, it is obtained by integrating out massive degrees of freedom starting from the full action of the β-deformed theory. As the latter has only $\mathcal{N} = 1$ supersymmetry and thus only the F-terms in the effective action are protected. This includes the low-energy gauge coupling, but not the radius of compactification or the effective lattice spacing which certainly depend on D-terms.

To make a more convincing version of the above non-renormalisation argument we can focus on a limit in which the theory recovers sixteen supercharges at all length scales. As the Higgs branch theory has deformation parameter $\beta = 2\pi/n$, the microscopic lagrangian goes over to that of the $\mathcal{N} = 4$ theory in a limit $n \to \infty$. As the masses of BPS states are protected in the $\mathcal{N} = 4$ theory, one can argue that any quantum corrections to the classical mass formula are supressed by powers of $1/n$. The resulting classical formulae (4.2), (4.3) should then become exact in the large-n limit. Unfortunately, even this argument will not help us understand the proposed continuum limit. In particular, Limit I is a simultaneous large-n and strong coupling limit. Although corrections to the classical formulae which go like $1/n$ may be supressed, those which go like g^2/n are not and may be important.

The points raised above indicate that the existence and nature of the continuum limit depends on the detailed dynamics of the strongly coupled gauge theory. The main aim of this paper is to study the continuum limit in a controlled way. As suggested in [9] the first step is to perform an S-duality transformation a reinterpret the strongly coupled Higgs branch theory in terms of the S-dual confining phase. The S-dual of Limit I is,

Limit I: we consider the $\mathrm{U}(mn)$ theory with gauge coupling coupling $\tilde{g}^2$, zero vacuum angle $\tilde{\vartheta} = 0$ and deformation parameter $\tilde{\beta} = 8\pi^2 i/\tilde{g}^2 n$. We focus on the theory in the confining phase vacuum specified by non-zero VEVs (3.8). We take the limit $n \to \infty$, $\tilde{g}^2 \to 0$ while holding m and $\tilde{g}^2 n$ (and thus $\tilde{\beta}$) fixed. We also hold the VEVs $|\tilde{\alpha}_1|$ and $|\tilde{\alpha}_2|$ fixed.

Thus we see that S-duality has three notable effects. First, because of the non-trivial modular transformation of the scalar VEVs, the new limit is one where the dimensionful parameters $\tilde{\alpha}_i$ labelling the vacuum are held fixed. Similarly, although the deformation parameter $\beta = 2\pi/n$ goes to zero in Limit I, the dual parameter $\tilde{\beta} = 8\pi^2 i/\tilde{g}^2 n$ remains fixed in the S-dual Limit I. This already illustrates one of the points made above: at fixed $\tilde{g}^2 n$ the S-dual theory only has $\mathcal{N} = 1$ supersymmetry in the UV. Thus the naive argument that $\mathcal{N} = 4$ supersymmetry is automatically recovered in Limit I simply because β goes to zero is wrong! Finally we see that S-duality maps a strong coupling limit of the Higgs phase theory to a 't Hooft limit of a confining phase theory. As the confinement is only partial, standard large-N scaling argument do not immediately apply. In section 9, we will reconsider this issue in detail in the context of deconstructing double-scaled LST.

The S-duality transformation also raises a puzzling issue about the effective lattice spacing in the confining phase. As in the Higgs phase, the scale invariance of the UV theory is broken only by the non-zero scalar vacuum expectation values (3.8). Thus we would expect the theory exhibit approximate conformal invariance above the energy scale,

$$E \sim \max[\tilde{\alpha}_1, \tilde{\alpha}_2] \,. \tag{5.3}$$

On the other hand, from the point of view of deconstruction, we would expect the scale where four-dimensional conformal invariance is recovered to be set by the smallest lattice spacing $\varepsilon = \min[\varepsilon_1, \varepsilon_2]$. Rewriting the classical formula (4.2) in terms of the confining phase variables, this corresponds to an energy scale,

$$E \sim \varepsilon^{-1} = \left(\frac{4\pi}{\tilde{g}^2}\right) \max[\tilde{\alpha}_1, \tilde{\alpha}_2] \,. \tag{5.4}$$

This discrepancy has an obvious origin. On the Higgs branch the lattice spacing is set by the masses of the heaviest W-boson. Assuming the classical mass formula is valid, we expect this state to become very massive in the strong-coupling limit. The corresponding lattice spacing would then go to zero. From the point of view of the confining phase this state is a magnetic monopole which is very heavy for small values of the dual coupling. On the other hand we might expect that, as in the underlying $\mathcal{N}=4$ theory, the scale at which conformal invariance is broken is actually set by the elementary electric degrees of freedom rather than the magnetic ones. These two possibilities lead to the two different formulae (5.3) and (5.4).

In the following we will clarify the issues raised above by explicit calculation. Specifically, if we choose $\tilde{g}^2 n \gg 1$, then the proposed continuum limit lies in a regime where we can hope to study it directly using the AdS/CFT correspondence. As the deformation parameter $\tilde{\beta}=8\pi^2 i/\tilde{g}^2 n$ is then small, we can think about the conformal theory as a small deformation of the $\mathcal{N}=4$ theory. On the other hand, as the 't Hooft coupling is large, the latter is well described by IIB supergravity on $AdS_5\times S^5$. The deformation can be incorporated by introducing a source for the corresponding supergravity field on the boundary of AdS_5. The main aim of the next two sections will be to construct an explict AdS dual for the β-deformed theory on its confining branch.

6. String theory dual of the Higgs branch

In this section we will construct the string dual of the β-deformed theory. First we review the embedding of the weakly-coupled Higgs branch in IIB string theory discussed in [9]. We then move on to consider the AdS dual of the strongly coupled Higgs branch theory. Finally, in the next section, we will use IIB S-duality to find the corresponding dual for the confining phase theory.

6.1 Weak coupling analysis

It is straightforward to embed the β-deformed theory in string theory [9] starting from the standard realization of $\mathcal{N}=4$ SUSY Yang-Mills theory with gauge group $\mathrm{U}(N)$ on the world-volume of a stack of N D3 branes in type-IIB string theory. As usual, the string coupling is related to the four-dimensional gauge coupling as $g_s=g^2/4\pi$. For convenience we will set the field theory vacuum angle ϑ (which is proportional to the background value of the RR scalar $C_{(0)}$) to zero in the following. In this subsection, we will initially work at small 't Hooft coupling $g^2N=4\pi g_s N\ll 1$, so that the gauge theory on the branes is weakly coupled.

The β-deformation is introduced by turning on a particular background for the complex threeform field-strength,

$$G_{(3)}=F_{(3)}-\tau H_{(3)} \tag{6.1}$$

which will be described in more detail in subsection 6.2 below.[7] Here $F_{(3)}$ and $H_{(3)}$ are the RR and NS three-form field strengths respectively. For small deformations the field strength is proportional to the deformation parameter β with a real constant of proportionality.

[7] At the linearized level, the relation between background supergravity fields and deformations of the $\mathcal{N}=4$ theory on the brane in weakly coupled string theory is essentially the same as the standard dictionary between SUGRA fields and chiral primary operators provided by the AdS/CFT correspondence [36].

Thus the theory with $\beta = 2\pi/n$ corresponds to a non-zero field strength for $F_{(3)}$ of order $1/n$. For large-n the deformation can be treated as a small perturbation and the back-reaction of this field strength on the geometry can be neglected.

At weak coupling, the Higgs branch which appears for $\beta = 2\pi/n$ corresponds to configurations where the D3 branes polarize into m D5 branes wrapped on a torus in the transverse R^6 [9]. We define convenient complex combinations of the coordinates, x_m with $m = 4, \ldots, 9$, in the six transverse dimensions,

$$z_1 = x_4 + ix_7 = \rho_1 e^{i\psi_1} \qquad z_2 = x_5 + ix_8 = \rho_2 e^{i\psi_2} \qquad z_3 = x_6 + ix_9 = \rho_3 e^{i\psi_3}\,. \tag{6.2}$$

In the vacuum (3.3) with $\alpha_3 = 0$, the D5 branes are wrapped on a rectangular torus $T^2(r_1, r_2)$ defined in terms of these coordinates by the equations,

$$\rho_1 = r_1 = |\alpha_1|(2\pi\alpha') \qquad \rho_2 = r_2 = |\alpha_2|(2\pi\alpha') \qquad \rho_3 = 0\,. \tag{6.3}$$

This configuration is energetically stable because of the non-zero background flux.

The configuration of wrapped D5's also carries a total of N units of D3 brane charge which can be realized as a constant non-zero background for the gauge-invariant combination,

$$\mathcal{F} = F - \frac{1}{2\pi\alpha'} B_{\rm NS} \tag{6.4}$$

where F is the world-volume gauge field-strength in the central U(1) of U(m) and $B_{\rm NS}$ is the Neveu-Schwarz two-form potential. The presence of N D3 branes then implies,

$$\int_{T^2} \mathcal{F} = 2\pi n\,. \tag{6.5}$$

We introduce flat coordinates χ_1 and χ_2 on the torus with $0 \leq \chi_1 \leq 2\pi r_1$ and $0 \leq \chi_2 \leq 2\pi r_2$. In this basis the metric on the torus is the flat one $g_{ab} = \delta_{ab}$. Choosing a gauge $F = 0$, the two-form field can be written as,

$$B_{\rm NS} = \frac{n}{2\pi r_1 r_2}\,(2\pi\alpha')\, d\chi_1 \wedge d\chi_2 = \frac{n}{2\pi|\alpha_1||\alpha_2|}\,\frac{1}{(2\pi\alpha')}\, d\chi_1 \wedge d\chi_2\,. \tag{6.6}$$

As usual the presence of $B_{\rm NS}$ introduces non-commutivity in the T^2 component of the $D5$ world volume. Standard arguments imply that the low-energy gauge theory on the $D5$ brane worldvolume six-dimensional non-commutative Yang-Mills theory with sixteen supercharges. Indeed one can check [8, 9] that this coincides precisely with the classical effective theory described in section 4, with parameters exactly as given in (4.3), (4.4).

6.2 Strong coupling analysis

In the previous subsection we reviewed the IIB brane configuration which realizes the weakly-coupled Higgs branch for $n \gg 1$ and $g^2N \ll 1$. In this subsection we will use the ideas of [15] to follow this configuration to the regime of strong 't Hooft coupling $g^2N \gg 1$. As before we have $N = mn$ and choose $n \gg 1$. In this case the Higgs branch deformation parameter $\beta = 2\pi/n$ is small and we will work perturbatively in $|\beta|$. We start by considering the undeformed $\mathcal{N}=4$ theory with $\beta=0$. At large 't Hooft coupling this is dual to the near horizon geometry of N D3 branes. The corresponding IIB supergravity background has the general form:

$$\begin{aligned}
ds^2 &= \mathcal{Z}^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + \mathcal{Z}^{+\frac{1}{2}}dx_m dx_m \\
F_5 &= d\chi_4 + \star d\chi_4 \\
\chi_4 &= \frac{1}{g_s\mathcal{Z}}dx_0\wedge dx_1\wedge dx_2\wedge dx_3 \\
e^\phi &= g_s \qquad C_{(0)} = \frac{\vartheta}{2\pi}
\end{aligned} \tag{6.7}$$

The coordinate indices x_μ, with $\mu=0,1,2,3$, are the four spacetime coordinates parallel to the branes and x_m, with $m=4,5,\ldots,9$ denote the six transverse directions. In the above χ_4 is the RR four-form potential and F_5 is the corresponding self-dual five-form field strength. Here $\mathcal{Z}$ denotes a harmonic function of the radial coordinate $r=\sqrt{x_m x_m}$.

The $\mathcal{N}=4$ theory with gauge group SU(N) at its conformal point is dual to the near horizon geometry of N coincident D3 branes. This corresponds to the choice, $\mathcal{Z}_{\rm UV}=\mathcal{R}^4/r^4$ in (6.7) with $\mathcal{R}^4=4\pi g_s N\alpha'^2$. The resulting geometry is precisely $AdS_5\times S^5$ with radius $\mathcal{R}$ for both factors. Gauge theory and string theory parameters are related according to,

$$g^2 = 4\pi g_s \qquad g^2N = \frac{\mathcal{R}^4}{\alpha'^2} \tag{6.8}$$

and the classical supergravity approximation is valid for our chosen regime $g^2\ll 1$, $g^2N\gg 1$.

As in the previous section, we are interested is a configuration where the N D3 branes are uniformly distributed on the torus $T^2(r_1,r_2)$ defined in (6.3). In $\beta=0$ case, this corresponds to a particular point on the Coulomb branch of the $\mathcal{N}=4$ theory. The corresponding near-horizon geometry takes the form (6.7) with the harmonic function $\mathcal{Z}(r)$ given as [37]

$$\mathcal{Z} = \frac{\mathcal{R}^4}{4\pi^2}\int_0^{2\pi}d\psi_1\int_0^{2\pi}d\psi_2\,\frac{1}{D(\psi_1,\psi_2)^2} \tag{6.9}$$

with,

$$D(\psi_1,\psi_2) = \rho_1^2 - 2\rho_1 r_1\cos\psi_1 + r_1^2 + \rho_2^2 - 2\rho_2 r_2\cos\psi_1 + r_2^2 + \rho_3^2\,. \tag{6.10}$$

This geometry provides a dual description of the Coulomb branch theory at large 't Hooft coupling. Note that $\mathcal{Z}\to\mathcal{Z}_{\rm UV}=\mathcal{R}^4/r^4$ as $r\to\infty$ and so the geometry asymptotes to $AdS_5\times S^5$ for large r. This corresponds to the RG flow of the dual theory to the $\mathcal{N}=4$ superconformal fixed point in the UV.

The geometry also simplifies in the region very close to the branes, which corresponds to,

$$\rho_1 - r_1 \ll r_1 \qquad \rho_2 - r_2 \ll r_2 \qquad \rho_3 \ll r_1, r_2\,. \tag{6.11}$$

In this region the integral (6.9) is dominated by the neighbourhood of the singular point $\psi_1=\psi_2=0$ and the corresponding warp-factor reduces to,

$$\mathcal{Z}_{\rm IR} = \frac{\mathcal{R}^4}{4\pi r_1 r_2 v^2} \tag{6.12}$$

where,

$$v = \sqrt{(\rho_1 - r_1)^2 + (\rho_2 - r_2)^2 + \rho_3^2} \tag{6.13}$$

is the distance from the toroidal shell of D3 branes.

As mentioned above, introducing the β-deformation corresponds to turning on a non-zero background for the complex three-form field strength $G_{(3)}$ proportional to the deformation parameter β. We will be mainly interested in the case of $|\beta| \ll 1$ which corresponds to introducing small field strength on the boundary of AdS_5. In the conformally invariant vacuum, this leads to a small deformation of the $AdS_5 \times S^5$ geometry which can be constructed order by order in $|\beta|$. This program was carried out explicitly to second non-trivial order in [23]. As the deformed theory is $\mathcal{N} = 1$ superconformal invariant one finds a dual of the form $AdS_5 \times \tilde{S}^5$. Here $\tilde{S}^5$ denotes a small deformation of S^5. The resulting perturbation away from the the round metric on S^5 reflects the fact that the SU(4) R-symmetry of the $\mathcal{N} = 4$ theory is broken to $\mathrm{U}(1)^3$ for non-zero β.

In the present case we will only need to work to linear order in the deformation. The resulting three-form flux can be deduced from the analysis of Graña and Polchinski [22]. In fact we will only need the "self-dual" component of the field-strength,

$$G^+_{(3)} = G_{(3)} + i \star_6 G_{(3)} \tag{6.14}$$

where $\star_6$ denotes the Hodge dual in the transverse R^6 parameterized by coordinates x_m appearing in (6.7) with the flat metric δ_{mn}. At linear order the resulting flux can be written as, $G^+_{(3)} = (i\rho\beta/3g_s)\mathcal{Z}(r)d\omega_2$ where ρ is a real constant we have not determined and ω_2 is a two-form which is conveniently written in terms of the complex coordinates introduced in (6.2) above. Explicitly we have,

$$\omega_2 = z_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + z_2 z_3 d\bar{z}_2 \wedge d\bar{z}_3 + z_3 z_1 d\bar{z}_3 \wedge d\bar{z}_1 \, . \tag{6.15}$$

This result holds in any asymptotically AdS background of the form (6.7) and we will apply it with the warp factor $\mathcal{Z}$ chosen as in (6.9). The criterion for a small deformation of the $AdS_5 \times S^5$ geometry is $|\beta| \ll 1$, which matches the condition for a small deformation of the $\mathcal{N} = 4$ theory on the boundary. In the following, we will take account of this non-zero three-form background and its effect on probe branes in the bulk. However, as we are working to linear order in $|\beta|$, we will not need to consider the back-reaction on the geometry explicitly which appears at order $|\beta|^2$.

We are now ready to construct the AdS dual of the Higgs branch theory. As at weak coupling our starting point is m D5 branes wrapped on the torus $T^2[r_1, r_2]$. The torus also carries N units of D3 brane charge. As above, the backreaction of the D3 branes leads to a near-horizon geometry of the form (6.7) with the non-trivial warp factor (6.9). In a complete treatment of the full system we would need to account for the backreaction of both the D3 branes and the wrapped D5 branes. This would be a very hard problem. Fortunately, as in [15], we may choose work in a regime of parameters where the problem simplifies.

To illustrate this we consider D5 branes embedded in the $AdS_5 \times S^5$ geometry corresponding to the large-r warp factor $\mathcal{Z}_{\rm UV}$ defined above. In this metric, the torus $T^2[r_1, r_2]$, defined in (6.3), is located at fixed radial position $r = \bar{r} = \sqrt{r_1^2 + r_2^2}$ in AdS_5. Our probe configuration becomes m D5 branes located at $r = \bar{r}$ and wrapped on a torus $T^2 \subset S^5$. The area of the torus is of order $\mathcal{R}^2 = \sqrt{4\pi g_s N}\alpha'$. The average density of D3 brane charge is therefore $\sigma_3 \sim n/\mathcal{R}^2$. This gives,

$$\sigma_3 \sim \sqrt{\frac{n}{g_s m}}\frac{1}{\alpha'}\,. \tag{6.16}$$

It follows that provided $g_s/n \ll m$, the energy density of D3 branes is large in string units and dominates that of D5 branes. This condition is trivially satisfied in our chosen region of parameter space

The above argument suggests that, for $g_s/n \ll m$, it is legitimate to treat the D5 branes as probe branes wrapped on a torus $T^2 \subset S^5$ in the Coulomb branch geometry (6.7), (6.9). The first obvious test of this reasoning is to check that such a configuration is stable. In the dual field theory, the Higgs branch only exists for special values of the deformation parameter. The brane configuration should therefore be stable only for the corresponding values of the complex three-form flux. As the complex moduli α_1 and α_2 are flat directions of the potential for $\beta = 2\pi/n$, the resulting configuration should then be stable for all values of the radii r_1 and r_2. In appendix A we perform a probe calculation using the Dirac-Born-Infeld action of the D5 brane and verify these expected properties.

In conclusion, the AdS dual of the Higgs branch consists of m wrapped D5 branes in the Coulomb branch geometry. In addition a small background flux for the RR threeform is present which has the role of stabilizing the the wrapped branes. As discussed above the backreaction of the D5 branes on the geometry can be consistently neglected almost everywhere. As in [15], the only exception to this is a thin layer close to the branes where the expansion of the metric in two directions transverse to the D3s and parallel to the D5s effectively dilutes the D3 brane charge. In this region the fivebrane charge effectively dominates. The SUGRA solution in this regime should match onto the near-horizon geometry of m flat D5 branes with world-volume $B_{\rm NS}$.

Following Polchinski and Strassler, our approximate solution is obtained by patching together the solution in the two regions described above. Thus we compare the Coulomb branch solution (6.7) for $\mathcal{Z} = \mathcal{Z}_{\rm IR}$ given above with the UV behaviour of the D5 brane background (2.5). In the limit $u \gg a^{-1}$ we have $\bar{h}(u) \simeq 1/a^2u^2$ and (2.5) can be written as,

$$\begin{aligned}
ds^2 &= \alpha'\left(\frac{s}{\sqrt{bg_s m}}\right)\eta_{\mu\nu}dy^\mu dy^\nu + \frac{1}{\alpha'}\left(\frac{\sqrt{bg_s m}}{s}\right)dy_m dy_m \\
B_{\rm NS} &= \frac{b}{\alpha'}\,dy_4\wedge dy_5 \\
\exp(2\phi) &= g_s^2 \\
\chi_4 &= \frac{1}{g_s}\left(\frac{s^2}{bg_s m}\right)dy_0\wedge dy_1\wedge dy_2\wedge dy_3\,.
\end{aligned} \tag{6.17}$$

Remarkably we find that these two metrics match exactly if we identify the coordinates as $y_\mu = x_\mu$ for $\mu = 0,1,2,3$ and $y_m = M_{mn}x_n + N_{mn}$, where M is an SO(6) rotation matrix and N_{mn} is another constant matrix, for $m,n = 4,5,\cdots,9$. This implies that the radial coordinate $s = \sqrt{y_6^2 + \cdots + y_9^2}$ is identified with the distance v from the branes. The dilaton and five-form solutions also match and the constant $B_{\rm NS}$ field in (6.17) can be removed with a gauge transformation. Comparing the coefficients appearing in the metric we find that,

$$b = \frac{\alpha'^2 n}{r_1 r_2} = \frac{n}{4\pi^2|\alpha_1||\alpha_1|}\,. \tag{6.18}$$

The resulting solution can be written as,

$$\begin{aligned} ds^2 &= \mathcal{Z}_{\rm IR}^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + \mathcal{Z}_{\rm IR}^{+\frac{1}{2}}\left(\frac{v^2}{v_{cr}^2+v^2}\right)[dx_4^2+dx_5^2] + \mathcal{Z}_{\rm IR}^{+\frac{1}{2}}[dv^2+v^2d\Omega_3^2] \\ B_{\rm NS} &= \frac{b}{\alpha'}\left(\frac{v^2}{v_{cr}^2+v^2}\right)\; dx_4\wedge dx_5 \\ e^{2\phi} &= g_s^2\left(\frac{v^2}{v_{cr}^2+v^2}\right) \\ \chi_4 &= \frac{1}{g_s\mathcal{Z}_{\rm IR}}dx_0\wedge dx_1\wedge dx_2\wedge dx_3 \end{aligned} \tag{6.19}$$

where $\mathcal{Z}_{\rm IR}$ is given in (6.12) and the crossover occurs at

$$v_{cr} = \sqrt{\frac{g_s m r_1 r_2}{n}}\,. \tag{6.20}$$

The proceedure of patching the two solutions together makes sense provided we can find a range of values where the two asymptotic forms used are simultaneously valid. This is equivalent to the condition $v_{cr} \ll r_1$, r_2 which holds provided $g_s m/n \ll 1$. The latter condition is trivially satisfied in the limit of interest (large-n with $g_s \ll 1$). Finally, the toroidal compactification of the worldvolume theory is implimented by a periodic identification of the coordinates in the x_4 and x_5 directions,

$$x_4 \sim x_4 + 2\pi r_1 \qquad x_5 \sim x_5 + 2\pi r_2\,. \tag{6.21}$$

Thus we find a geometry which interpolates between $AdS_5\times S^5$ far away from the wrapped branes and a geometry which coincides with the holographic dual of the decoupled five-brane theory $\mathcal{T}[M_s, g_s]$. Using the matching (6.18) we can identify the parameters of the five-brane theory in terms of those of the four-dimensional gauge theory. As always we have $g_s = g^2/4\pi$ and in addition we find,

$$M_s = \sqrt{\frac{|\alpha_1||\alpha_2|}{g^2 n}} \tag{6.22}$$

at low energies the fivebrane theory $\mathcal{T}[M_s, g_s]$ reduces to a non-commutative six dimensional gauge theory with $G_6 = 1/M_s$ and $\theta = 1/8\pi^2 g_s M_s^2$. This precisely reproduces the values of these parameters (4.4) computed in weakly coupled field theory. As at weak coupling, the compactification radii of the worldvolume theory should be measured in the open string metric. As in [9], converting the closed string radii r_1 and r_2 to the corresponding lengths in the open string metric reproduces the classical formula (4.3) for the radii R_1 and R_2.

We can now try to interpret our solution as describing an RG flow between four-dimensional superconformal field theory in the UV and six dimensional non-commutative gauge theory in the IR. In the UV region $r \gg \bar{r}$ where conformal invariance is restored, the UV/IR connection provides a relation between the radial dependence of the supergravity solution and RG flow in the dual field theory. As the deviations from the AdS_5 geometry become significant at $r \sim \bar{r}$ we identify the corresponding energy scale,

$$\Lambda = \frac{\bar{r}}{\sqrt{g_s N \alpha'}} \sim \frac{1}{\sqrt{g^2 N}} \max\left[|\alpha_1|, |\alpha_2|\right] \tag{6.23}$$

with the scale in the dual field theory below which the breaking of conformal symmetry becomes significant. Above this mass-scale, the theory looks like a four-dimensional conformal field theory with no trace of six dimensional behaviour. Equivalently, this scale is the effective momentum-space cutoff on the low-energy six-dimensional theory. In the language of deconstruction it is natural to identify this with the inverse of the smaller lattice-spacing $\varepsilon = \min[\varepsilon_1, \varepsilon_2]$. At weak coupling the corresponding scale is given by (4.2) as,

$$\Lambda_{\mathrm{WC}} = \frac{1}{\varepsilon} = \max\left[|\alpha_1|, |\alpha_2|\right]. \tag{6.24}$$

Thus already we see a significant (but quite familiar) discrepancy between weak and strong coupling. This indicates a non-trivial renormalization of the lattice spacing of the sort discussed in section 1. The discrepancy will become even more severe as we go to the regime with $g^2 \gg 1$ where the continuum limit is supposed to lie.

7. String theory dual of the confining branch

The field theory S-duality reviewed in section 2 predicts that the β-deformed theory with gauge coupling $\tilde{g}^2$, zero vacuum angle and deformation parameter $\tilde{\beta} = 8\pi^2 i/\tilde{g}^2 n$ has a confining branch. We will focus on the vacua with non-zero chiral VEVs labelled, as in (3.8), by complex parameters $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ (with $\tilde{\alpha}_3 = 0$).

In the string theory realization of the β-deformed theory, it is easy to see that field theory S-duality coincides exactly with the S-duality of the IIB string. Apart from the usual $\mathrm{SL}(2,\mathbb{Z})$ action on the complexified string coupling, $\mathrm{SL}(2,\mathbb{Z})$ transformations act as linear transformations on the the RR and NS three-form field strengths $F_{(3)}$ and $H_{(3)}$. Equivalently, the complexified three-form field strength $G_{(3)}$ transforms with holomorphic weight $(-1,0)$ under $\mathrm{SL}(2,\mathbb{Z})$. This matches the transformation of the deformation parameter β under field theory S-duality.

We can now use IIB S-duality to find the string theory dual of the confining phase vacuum. Starting with the background value for the Ramond-Ramond field strength $F_{(3)}$ corresponding to $\beta = 2\pi/n$, we end up with a non-zero background for the Neveu-Schwarz field strength $H_{(3)}$ corresponding to the dual theory with $\tilde{\beta} = 8\pi^2 i/\tilde{g}^2 n$. Similarly, starting with m D5 branes wrapped on $T^2(r_1, r_2)$ we end up with m NS 5 branes wrapped on the same torus. In terms of the S-dual variables we have, $r_1 = |\tilde{\alpha}_1|(2\pi\tilde{\alpha}')$ and $r_2 = |\tilde{\alpha}_2|(2\pi\tilde{\alpha}')$. Finally the N units of D3 brane charge are invariant under S-duality and correspond to a constant background value of the Ramond-Ramond two-form potential B_{RR} in the torus component of the NS5-brane world volume. This configuration corresponds to the confining phase vacuum of the dual theory.

It is important to note that, unlike the Higgs branch configuration, there is no regime in which which the system is simply described by toroidally wrapped branes in flat space. At small 't Hooft coupling $\tilde{g}^2 n \ll 1$ the deformation corresponding to $\tilde{\beta} = 8\pi^2 i/\tilde{g}^2 n$ becomes large and we cannot ignore its effect on the geometry. On the other hand, for $\tilde{g}^2 n \gg 1$, we have a small deformation of the strongly coupled $\mathcal{N} = 4$ theory. In this case, as for the Higgs vacuum, we should find a supergravity dual which asymptotes to $AdS_5 \times \tilde{S}^5$ for large values of the radial coordinate. In fact we can simply apply the S-duality transformation to the geometry describing the Higgs phase vacuum derived in the previous section. The S-dual configuration will consist of m NS5 branes wrapped on a torus $T^2 \subset S^5$ and located at fixed radial position $\bar{r} = \sqrt{r_1^2 + r_2^2}$ in the near horizon geometry of the D3 branes.

The supergravity solution describing the confining phase vacuum is obtained by applying the transformation (2.8) to the Higgs branch solution. Under this transformation, the UV component of the solution (6.7), which describes a point on the Coulomb branch of the $\mathcal{N} = 4$ theory is self-dual. On the other hand the geometry near the wrapped branes (6.19) transforms non-trivially and becomes,

$$
\begin{aligned}
d\tilde{s}^2 &= \mathcal{Z}_{\mathrm{IR}}^{-\frac{1}{2}} \left(\frac{v_{cr}^2 + v^2}{v^2}\right)^{\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \mathcal{Z}_{\mathrm{IR}}^{+\frac{1}{2}} \left(\frac{v^2}{v_{cr}^2 + v^2}\right)^{\frac{1}{2}} [dx_4^2 + dx_5^2] + \\
&\quad + \mathcal{Z}_{\mathrm{IR}}^{+\frac{1}{2}} \left(\frac{v_{cr}^2 + v^2}{v^2}\right)^{\frac{1}{2}} [dv^2 + v^2 d\Omega_3^2] \\
B_{\mathrm{RR}} &= \frac{b}{\tilde{\alpha}'} \left(\frac{v^2}{v_{cr}^2 + v^2}\right) dx_4 \wedge dx_5 \\
e^{2\tilde{\phi}} &= \tilde{g}_s^2 \left(\frac{v_{cr}^2 + v^2}{v^2}\right) \\
\chi_4 &= \frac{1}{\tilde{g}_s \mathcal{Z}_{\mathrm{IR}}} dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}
\tag{7.1}
$$

As above we have $\mathcal{Z}_{\mathrm{IR}} = \mathcal{R}^4/4\pi r_1 r_2 v^2$ where, in terms of the S-dual variables, $\mathcal{R}^4 = 4\pi \tilde{g}_s N \tilde{\alpha}'^2$. As in the Higgs branch solution we have the identifications,

$$
x_4 \sim x_4 + 2\pi r_1 \qquad x_5 \sim x_5 + 2\pi r_2 \tag{7.2}
$$

The resulting geometry asymptotes to $AdS_5 \times S^5$ near the boundary and undergoes a smooth transition to the near-horizon geometry of m flat NS5 branes with background B_{RR} given in (2.9). In particular, for distance from the brane less than the cross-over distance,

$$
v_{cr} = \sqrt{\frac{m r_1 r_2}{\tilde{g}_s n}} \tag{7.3}
$$

we enter the infinite throat region of the NS5 brane solution described by the metric (2.14). Eqn (7.2) implies that the coordinates Y_4 and Y_5 along the NS5 branes appearing in (2.14) are identified as,

$$
Y_4 \sim Y_4 + 2\pi L_1 \qquad Y_5 \sim Y_5 + 2\pi L_2 \tag{7.4}
$$

where,

$$L_i = \mathcal{Z}_{IR}^{\frac{1}{4}} \left(\frac{v}{v_{cr}}\right)^{\frac{1}{4}} r_i = \left(\frac{\tilde{g}_s n \tilde{\alpha}'}{r_1 r_2}\right)^{\frac{1}{2}} r_i \tag{7.5}$$

for $i = 1, 2$. As before, the patching condition for matching the asymptotic forms of the two regions is $v_{cr} \ll r_1$, r_2. This is satisfied provided we work at sufficiently large 't Hooft coupling, $\tilde{g}_s n \gg m$.

The geometry near the branes coincides with holographic dual of the fivebrane world-volume theory $\mathcal{T}[M_s, g_s]$. Once again we can identify the parameters of the five-brane theory in terms of those of the four-dimensional gauge theory. We find $g_s = 4\pi/\tilde{g}^2$ and,

$$M_s = \sqrt{\frac{|\tilde{\alpha}_1||\tilde{\alpha}_2|}{\tilde{g}^2 n}} \tag{7.6}$$

at low energies the fivebrane theory $\mathcal{T}[M_s, g_s]$ reduces to a six-dimensional gauge theory with $G_6 = 1/M_s$. This result for G_6 agrees with the field theory result obtained by rewriting the first equality in (4.4) in terms of the S-dual confining phase variables.

As for the Higgs branch solution we can attempt to understand our solution in terms of an RG flow between four-dimensional superconformal field theory in the UV and six dimensional theory in the IR. In the UV region $r \gg \bar{r}$ where conformal invariance is restored, the UV/IR connection provides a relation between the radial dependence of the supergravity solution and RG flow in the dual field theory. As the deviations from the AdS_5 geometry become significant at $r \sim \bar{r}$ we identify the corresponding energy scale,

$$\Lambda = \frac{\bar{r}}{\sqrt{\tilde{g}_s N}\tilde{\alpha'}} \sim \frac{1}{\sqrt{\tilde{g}^2 N}} \max\left[|\tilde{\alpha}_1|, |\tilde{\alpha}_2|\right]. \tag{7.7}$$

This is to be contrasted with the field theory result (rewritten in the S-dual variables)

$$\Lambda_{\rm WC} = \frac{1}{\varepsilon} = \left(\frac{4\pi}{\tilde{g}^2}\right) \max\left[|\tilde{\alpha}_1|, |\tilde{\alpha}_2|\right]. \tag{7.8}$$

Once again there is a striking discrepancy between the weak coupling and strong coupling results indicating non-trivial renormalisation of the lattice spacing in the language of deconstruction. In our chosen region of parameters it seems that the effective UV cut-off on the six-dimensional theory is not larger than the dynamical scale of the theory M_s. This is quite striking because we are in exactly the regime of parameters which should be close to the continuum limit predicted on the basis of weak coupling arguments in section 3. It seems that no such continuum limit exists.

Despite the above, we still find an interesting result when we take the 't Hooft limit $n \to \infty$, $\tilde{g}^2 \to 0$ with $\tilde{g}^2 n$ fixed. The cross-over scale v_{cr} remains fixed in this limit as does the dynamical scale M_s. The effective string coupling $\exp(\tilde{\phi})$ vanishes uniformly for all $v \geq v_{cr}$. The region of the geometry where the coupling remains non-zero is in the tube $v \ll v_{cr}$. In fact, to find a non-zero effective coupling we need to look at the region where $v \sim \tilde{g}_s$ as $\tilde{g}_s \to 0$. This is precisely the limit discussed in section 4, where the full solution (2.9) reduces to the near horizon geometry of m NS five-branes with no additional fields turned on. The theory in this region is identical to Little String Theory and it is completely decoupled from all of the other states in the theory. The LST in question is compactified on a torus of radii L_1 and L_2 given in (7.5) above. As explained at the end of section 2 we must convert these lengths into field theory units to find compactification radii,

$$R_1 = \left(\frac{\hat{M}_s}{M_s}\right) L_2 = \frac{\tilde{g}^2 n}{8\pi^2|\tilde{\alpha}_1|} \qquad R_1 = \left(\frac{\hat{M}_s}{M_s}\right) L_1 = \frac{\tilde{g}^2 n}{8\pi^2|\tilde{\alpha}_2|} \tag{7.9}$$

where $\hat{M}_s = 1/\sqrt{16\pi^3\tilde{\alpha}'}$. This agrees exactly with the weak coupling formula (4.3) written in terms of the confining phase variables.

8. The continuum limit revisited

The results of the previous section establish a modified version of the deconstruction conjecture. Summarizing the above, we will now state the main result starting from the strongly-coupled confining phase of the β-deformed theory. For brevity, we will now drop all tildes on confining phase quantities.

We consider the β-deformed theory with gauge group $\mathrm{U}(mn)$, gauge coupling g^2, deformation parameter $\beta = 8\pi^2 i/g^2 n$ and zero vacuum angle. This theory has a moduli space of vacua where the $\mathrm{U}(mn)$ gauge symmetry is confined down to a $\mathrm{U}(m)$ subgroup. We will consider the vacuum state specified by the moduli,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_1^n\right]\right\rangle = \alpha_1^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_2^n\right]\right\rangle = \alpha_2^n \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_3^n\right]\right\rangle = 0\,. \tag{8.1}$$

Result: in the limit $n \to \infty$, $g^2 \to 0$ with $g^2 n$, m, α_1 and α_2 fixed, the interacting sector of the theory is equivalent to type-IIB Little String Theory with low energy gauge group $\mathrm{SU}(m)$ and string mass-scale,

$$M_s = \sqrt{\frac{|\alpha_1||\alpha_2|}{g^2 n}} \tag{8.2}$$

compactified to four dimensions on a torus of radii,

$$R_1 = \frac{g^2 n}{8\pi^2|\alpha_1|} \qquad R_2 = \frac{g^2 n}{8\pi^2|\alpha_2|} \tag{8.3}$$

with supersymmetry-preserving boundary conditions. In addition to the LST degrees of freedom, the β-deformed theory contains additional massive and massless states which are completely decoupled in this limit.

In the previous section we were able to demonstrate this result explicitly for the case $g^2 n \gg m$. This corresponds to the case where the compactification torus is much larger than the Little String length scale: $R_1 R_2 \gg M_s^{-2}$. The dual brane configuration continues to exist when this restriction is relaxed and the fact that fivebrane worldvolume decouples from gravity as the asymptotic string coupling goes to zero should be true generally. Thus we conjecture that the result holds for all values of $g^2 n$. In the more general case, the approximate supergravity solution obtained above is no longer valid. Provided the weaker condition $g^2 n \gg 1/m$ is satisfied, the UV behaviour of the theory should still be described by classical supergravity on $AdS_5 \times X_5$ for some five-manifold X_5 which is no longer a small deformation of the round S^5. Without further information, it would be hard to verify the more general conjecture directly.

We will now review the basic properties of Little String Theory on $R^{3,1} \times T^2$ and the extent to which they are correctly reproduced by our proposed deconstruction.

Symmetries: type-IIB Little String Theory has $\mathcal{N} = (1,1)$ super-Poincare symmetry in six dimensions which is broken to $\mathcal{N} = 4$ super-Poincare symmetry in four dimensions by compactification on T^2. The $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of translations on T^2 is also left unbroken and the corresponding momenta appear as central charges in the four-dimensional $\mathcal{N} = 4$ SUSY algebra. The low-energy gauge group is $\mathrm{SU}(m)$ and the theory also has an $\mathrm{SO}(4)$ global R-symmetry.

At first sight the symmetries of the β-deformed theory are quite different. At finite n the theory has only $\mathcal{N} = 1$ super-Poincare invariance in four-dimensions. The low-energy gauge group on the confining branch discussed above is $\mathrm{U}(m)$ and the theory has a $\mathrm{U}(1)^3$ R-symmetry broken to $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathrm{U}(1)$ in the vacuum (8.1). As in other deconstruction scenarios, the idea is that the finite n theory corresponds to a lattice regularisation of the higher-dimensional theory with the $\mathrm{U}(1) \times \mathrm{U}(1)$ of translations on T^2 broken to $\mathbb{Z}_n \times \mathbb{Z}_n$ corresponding to translations on an $n \times n$ lattice. The lattice also breaks the six-dimensional $\mathcal{N} = (1,1)$ supersymmetry down to $\mathcal{N} = 1$ in four dimensions. This is realised very explicitly in the weakly coupled Higgs branch regime of section 3.1 which corresponds to a very large lattice spacing.

The analysis of the strongly-coupled continuum limit given in the previous section showed that, rather than the lattice spacing going to zero, what really happens is that an interacting sector of the theory, which recovers the full symmetries of LST, decouples completely from the rest of the states in the theory. It is important to emphasise that the full theory, including the extra decoupled states, will respect some but not all of these symmetries. As $n \to \infty$ in the continuum limit, the $\mathbb{Z}_n \times \mathbb{Z}_n$ R-symmetry of the confining phase theory goes over to the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of translations on the torus. On the other hand, the full SUSY algebra of LST is only recovered in the interacting sector. The remaining unbroken $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ R-symmetry of the β-deformed theory corresponds to a subgroup of the $\mathrm{SO}(4)$ R-symmetry of LST. The supergravity analysis of the preceeding section shows that the full $\mathrm{SO}(4)$ symmetry is only recovered in the interacting sector of the theory.

Spectrum: the effective action for the massless modes of LST compactified on T^2 is $\mathcal{N} = 4$ SUSY Yang-Mills in four dimensions with gauge group $\mathrm{SU}(m)$ and gauge coupling $G_4^2 = 4\pi^2 M_s^2 R_1 R_2$. Using the conjectured identifications (8.2), we find $G_4^2 = 16\pi^2/g^2 n$ The exact effective action for the massless modes of the β-deformed theory on its confining branch is $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\mathrm{U}(m)$ and gauge coupling $g^2 n$. This is consistent with the conjecture because the $\mathcal{N} = 4$ theory has an exact $\mathrm{SL}(2, \mathbb{Z})$ duality which acts as $G_4^2 \to 16\pi^2/G_4^2$. The $\mathcal{N} = 4$ vector multiplet corresponding to the central $\mathrm{U}(1) \subset \mathrm{U}(m)$ should decouple in the proposed continuum limit.[8]

[8] As the theory contains only adjoint fields, the gauge boson of this multiplet is completely decoupled to start with. Note however that this is not true of the remaining fields in this multiplet at finite n.

The agreement of the massless fields means that, locally, the vacuum moduli space of the β-deformed theory is the same as that of compactified LST. Globally, however, the two moduli spaces differ at finite n. The confining branch of the β-deformed theory is the complex orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. This is a submanifold of a larger branch on which the unconfined gauge symmetry is $\mathrm{U}(1)^m$. The full branch is the m-fold symmetric product of the orbifold. On the other hand the moduli space of LST on T^2 is an m-fold symmetric product of $T^2 \times R^4$. In fact the two moduli spaces agree in the large-n continuum limit by exactly the same mechanism as in the standard deconstruction based on a quiver gauge group [7]. In particular one may think of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ as a cone over an $S^5/\mathbb{Z}_n \times \mathbb{Z}_n$ base which becomes very narrow in the large-n limit. The neighbourhood of a point on $C^3/\mathbb{Z}_n \times \mathbb{Z}_n$ goes over to $T^2 \times R^4$ in the large-n limit and the two moduli spaces agree.

LST compactified on the torus also has a spectrum of massive BPS states which preserve one half of the supersymmetry algebra. These include two towers of Kaluza-Klein states coming from compactification of the massless fields in six dimensions. In field theory, these states appear in the weakly-coupled Higgs branch regime of classical deconstruction. Our AdS analysis shows that they persist in the strongly coupled theory (at least for $g^2 n \gg 1$). LST also contains BPS strings in six-dimensions which can wind around either cycle of T^2 thereby producing two towers of BPS states in four dimensions. BPS saturated strings of the same tension appear as chromomagnetic flux tubes in the S-dual dual Higgs description of the confining phase theory. The Higgs phase theory reduces to a non-commutative six-dimensional gauge theory at low energies and the BPS strings can be understood as Yang-Mills instantons embedded in six-dimensions. The expected BPS winding modes should correspond to bound states in quantum mechanics on the moduli space of instantons.

More generally, the existence of BPS winding modes leads to an $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ T-duality group for LST compactified to four dimensions on a torus. One transformation which is particularly interesting coresponds to a simultaneous T-duality on both cycles of the torus. In the β-deformed theory this corresponds to a transformation which inverts the 't Hooft coupling $g^2 n$ as,

$$g^2 n \to \frac{16\pi^2}{g^2 n}\,. \tag{8.4}$$

As the symmetry relates small and large values of the 't Hooft coupling, it is a test of the strong form of the deconstruction conjecture, which applies for all values of $g^2 n$. In fact this duality is evident in the low-energy effective action and corresponds to an electric-magnetic duality transformation of the $\mathrm{U}(m)$ $\mathcal{N}=4$ effective theory of the massless modes. If the conjecture is correct this should also be an exact duality of the the full theory in the 't Hooft limit.

9. An application

So far we have focussed on deconstructing LST at the origin of its moduli space where the low energy gauge symmetry is $\mathrm{SU}(m)$. The holographic dual of this theory includes a linear background for the dilaton which means that the effective string coupling becomes large for small values of the radial coordinate. The presence of a strong coupling region makes it hard to perform reliable calculations of the observables of LST using the dual description. However, LST also has a moduli space of vacua on which the low-energy gauge group is broken to $\mathrm{U}(1)^{m-1}$. Following [26], it is possible to define a double-scaling limit of the theory on this branch which yields a weakly coupled holographic dual. In this section, we will review the resulting Double-Scaled Little String Theory (DSLST) and show how to extend our proposed deconstruction to this case.

As for the theory at the origin, DSLST is defined as a decoupling limit of the world volume theory on m D5 branes in type-IIB string theory. Before taking any limit, the low-energy theory on these branes is a six-dimensional $\mathrm{U}(m)$ gauge theory which includes four real adjoint scalar fields. Each field is an $m \times m$ matrix and, with an appropriate choice of normalisation, the eigenvalues specify the positions of the branes in their four transverse directions. We will consider a configuration where the m NS5 branes distributed symmetrically around a circle of radius r_0 in the transverse R^4. If we combine the four real scalar fields into two complex scalars A and B, this corresponds to choosing VEVs $\langle A \rangle = 0$ and $\langle B \rangle \sim r_0 U_{(m)}$. As above, $U_{(m)}$ denotes the $m \times m$ clock matrix. In this vacuum, the spectrum of the theory includes massive W-bosons corresponding to strings stretched between the D5 branes. These states have masses,

$$M_W^{(ab)} = \frac{1}{2\pi\alpha'}\, 2r_0 \sin\left(\frac{\pi}{m}|a-b|\right) \tag{9.1}$$

where $a, b = 1, 2, \ldots, m$ are integers labelling the branes. Note also that the global $\mathrm{SO}(4)$ symmetry of the theory is spontaneously broken to $\mathrm{SO}(2) \times \mathbb{Z}_m$ by the scalar VEVs.

Double-scaled Little String Theory is obtained by taking the limit $g_s \to \infty$, $r_0 \to 0$ with the mass scale $\bar{M} = 1/g_s r_0$ held fixed. As in the conventional definition of LST we also keep the six-dimensional gauge coupling, $\hat{G}_6 = \sqrt{16\pi^3\alpha' g_s}$ fixed. As in the basic case, we can reinterpret this limit by performing an S-duality transformation (2.1). Thus we start from a configuration of m NS5 branes uniformly distributed around a circle of radius r_0. The W-bosons (9.1) now correspond to D-strings stretched between the NS5 branes. The double-scaling limit is now $\tilde{g}_s \to 0$, $r_0 \to 0$ with $\hat{G}_6 = \sqrt{16\pi^3\tilde{\alpha}'}$ and $\bar{M} = \tilde{g}_s/r_0$ held fixed.

According to [26], this limit yields an exact superstring background of the form (see also [38]),

$$R^{5,1} \times \left(\frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right)/\mathbb{Z}_m \tag{9.2}$$

where the level of the two cosets appearing in the brackets is equal to m, the number of NS5 branes. The non-compact coset $\mathrm{SL}(2)/\mathrm{U}(1)$ corresponds to the two-dimensional black hole geometry or cigar. In an asymptotic region, far from the tip of the cigar, the background has a linear dilaton which matches that of the coincident NS5 solution, (2.14). Importantly, the string coupling is bounded above by its value at the tip of the cigar which is controlled by the ratio of the two mass scales defined above,

$$g_{\text{cigar}} \sim \frac{\bar{M}}{M_s}\,. \tag{9.3}$$

Thus, provided we choose this ratio to be small, the background can be studied reliably using string perturbation theory. Even better, the background (9.2) corresponds to an exactly solvable conformal field theory, so that tree-level string theory is fully tractable.

It is straightforward to adapt the proposal of the previous section to provide a deconstruction of DSLST. The details of the necessary modification are given in appendix B. As in the previous section we will drop all tildes on confining branch variables. We start from the β-deformed theory with $\beta = 8\pi i/g^2 n$ in the vacuum where the $\mathrm{U}(N)$ gauge group, with $N = mn$ is confined down to $\mathrm{U}(m)$ and the non-zero chiral operators are given by (8.1). The low-energy action for the massless degrees of freedom is precisely $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\mathrm{U}(m)$. We now move onto the Coulomb branch of the low-energy theory by introducing a non-zero VEV for the field Φ_3 which is written in gauge invariant form as,

$$\left\langle \frac{1}{N} \operatorname{Tr}\left[\Phi_3^N\right] \right\rangle = \mu^N \tag{9.4}$$

with all operators of the form $\operatorname{Tr}[\Phi_3^k]$ having zero VEV for $k < N$. Equation (9.4) together with (8.1) determines a particular vacuum of the theory where the $\mathrm{U}(mn)$ gauge group is confined down to $\mathrm{U}(1)^m$.

In the case $g^2 n \gg m$ which corresponds to a large compactification torus in string units, we can study the 't Hooft limit using AdS duality. The dual geometry again involves m wrapped NS5 branes in the Coulomb branch geometry (6.7), (6.9). The only difference is that the m branes are now wrapped on m distinct tori with small angular seperations on S^5. As we are ultimately interested in a limit where the typical separation, $d \sim \mu\tilde{\alpha}'$, between the fivebranes becomes small we can assume $d \ll v_{cr}$ where v_{cr} is the cross-over distance for the fivebrane solution given in (7.3). Instead of matching onto a solution of m coincident NS5 flat branes, we can now patch the solution onto the solution for m NS5 branes distributed uniformly around a circle of radius d. We can then scale μ in the 't Hooft limit so that the resulting decoupled theory is DSLST. The final result is,

Result: in the limit $n \to \infty$, $g^2 \to 0$, $\mu \to 0$ with $g^2 n$, m, α_1, α_2 and μn fixed, the interacting sector of the theory in this vacuum is equivalent to double-scaled Little String Theory with low energy gauge group $\mathrm{U}(1)^{m-1}$ and mass parameters,

$$M_s = \sqrt{\frac{|\alpha_1||\alpha_2|}{g^2 n}} \qquad \bar{M} = \frac{2\pi|\alpha_1||\alpha_2|}{|\mu|n} \tag{9.5}$$

compactified on a torus of radii R_1 and R_2 given in (8.3). In addition to the LST degrees of freedom, the β-deformed theory contains additional massive and massless states which are completely decoupled in this limit. As before our AdS analysis is only valid for large 't Hooft coupling $g^2 n \gg m$ and the full conjecture which applies for all values of $g^2 n$ is much harder to verify.

Putting everything together we find that, in the t' Hooft large-n limit described above, the interacting sector of the β-deformed theory is fully equivalent to the IIB string theory background,

$$R^{3,1} \times T^2 \times \left(\frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right)/\mathbb{Z}_m \tag{9.6}$$

where the IIB squared string-length α' is identified as $M_s/16\pi^3$ and the radii R_1 and R_2 of the compactification torus are given as in (8.3). As above both cosets have level m, and the dilaton is linear in the asymptotic region far from the tip of the cigar. The effective string coupling takes its maximum value (9.3) at the tip of the cigar. Thus if, we also choose $\bar{M} \ll M_s$, the dual string theory is weakly coupled. This means that in this corner of parameter space the interacting sector of β-deformed gauge theory can be solved exactly.

The tree-level spectrum and correlation functions of DSLST have been studied in detail in [26] and our first task is to understand these results in the context of the dual gauge theory. As a preliminary, it is useful to recall some basic facts about large-N gauge theories. We begin by considering an SU(N) gauge theory containing only adjoint fields in a phase where all coloured states are confined. In this case, the spectrum will consist entirely of colour-singlet glueballs. In the 't Hooft large-N limit, standard counting arguments suggest there should be an infinite tower of stable glueballs whose masses remain constant as $N \to \infty$. The effective three-point coupling constant governing the interactions of these states scales like $1/N$ and thus they become weakly-interacting in the large-N limit. The expected spectrum therefore resembles that of a closed string theory with effective string coupling constant $g_s^{\mathrm{eff}} \sim 1/N$. The mass scale of the lightest states in the spectrum is set by the inverse string tension.

In the present context we are not interested in a standard confining phase, but rather in a more exotic phase where a U(N) gauge group is confined down to a U$(1)^m$ subgroup. As the spectrum includes the gauge bosons of the unbroken gauge group there is no mass gap. The presence of an unconfined U$(1)^m$ gauge symmetry also changes the picture because the spectrum can now contain states which are electrically or magnetically charged with respect to this gauge symmetry. The exact S-duality of the low-energy theory interchanges these states and also inverts the low-energy gauge coupling: $G_4^2 \to 16\pi^2/G_4^2$. In our string theory construction, the electrically charged states correspond to D1 branes stretched between the NS5-branes with masses of order $M_W = M_s^2/\bar{M}$. There are also magnetically charged states corresponding to D3 branes stretched between the NS5 branes and wrapped on T^2 with masses of order $g^2 n M_W = R_1 R_2 M_s^3/\bar{M}$. Unlike the colour singlet states, there is no reason why these states should be weakly interacting at large-N. In particular these states couple to the massless U$(1)^m$ gauge fields with effective gauge coupling $16\pi^2/g^2 n$ which remains fixed in the 't Hooft limit.

The above discussion suggests that the large-N behaviour of a partially confined phase is quite different from a more conventional phase where the whole gauge group is confined. However, as these differences are due to the presence of states charged under the unconfined gauge symmetry they should disappear in a limit where we decouple the extra charged states. By choosing $\bar{M} \ll M_s$ we ensure that all charged states become very massive and we expect the remaining spectrum of colour singlets to resemble that of a conventional large-N confining gauge theory but with the strength of the residual interactions controlled by the ratio $\bar{M}/M_s$.

We can now compare these expectations with the analysis of the string theory background (9.2) for $\bar{M} \ll M_s$ given in [26, 27]. The first obvious point is that we have a dual description in terms of weakly coupled closed string theory in this regime with the effective string coupling controlled by the ratio $\bar{M}/M_s$ which matches our gauge theory expectations. The spectrum of string states includes a discrete set of states localised at the tip of the cigar. These include the expected massless states corresponding to the $m-1$ $\mathcal{N}=(1,1)$ vector multiplets of the low-energy gauge theory. The spectrum includes infinite towers of excited string states with a Hagedorn density at high energies. The tree-level S-matrix for these states exhibits Regge behaviour. These features match the expected behaviour of the dual gauge theory discussed above.

There are also other features of DSLST which are quite mysterious. In particular, the spectrum of DSLST also contains a continuum of states. These states are δ-function normalisable plane waves which scatter off the tip of the cigar. This is unexpected from the point of view of the dual gauge theory but is certainly an artifact of the large-N limit. If we work at fixed large $N = mn$ and fixed gauge coupling $m/n \ll g^2 \ll 1$, the dual geometry is no longer a cigar of infinite length. Instead, the NS5 brane throat opens out into an asymptotically AdS region at some fixed but large distance from the tip. There are no such δ-function normalisable states in AdS_5 and a generic plane wave solution in the throat will give rise to a non-normalizable solution in the AdS region. In this case it is natural to expect that the continuum of DSLST is replaced by a discrete spectrum.[9] The dual gauge theory provides a natural UV regulator for the cigar. Note that the finite-N geometry described above contains non-zero RR fields and is no longer an exactly solvable string background. However, at large 't Hooft coupling and for a large number of fivebranes $m \gg 1$, the curvature of the background is everywhere small and classical supergravity should be reliable. It would be interesting to investigate some of the other puzzling features of DSLST in this framework such as the limits on the existence of off-shell observables and the unusual zero-momentum behaviour discovered in [27].

The author would like to thank Ofer Aharony and Tim Hollowood for useful discussions. He would also like to thank IHES for its hospitality while this work was being completed.

A. The probe calculation

As a first step we will start by considering a q probe D5 branes each carrying p units of D3 brane charge and wrapped on the torus $T(r_1, r_2)$, defined in (6.3), in a geometry of the general form (6.7). If we choose the warp factor $\mathcal{Z}_{\rm UV} = \mathcal{R}^4/r^4$ then the branes are wrapped on a T^2 submanifold of S^5 located at fixed radial distance $r = \bar{r}$ in AdS_5. However, we will actually find that the stability conditions for the probe do not depend on the choice of warp factor.

[9] Interestingly the mass gap for the continuum modes is $M_s/\sqrt{m}$ which exactly matches the cut-off energy scale Λ given in (7.7) above. The latter is the characteristic energy for SUGRA states propagating in the transition region between the NS5 brane throat and the UV geometry. This is consistent with the fact that the continuum modes can propagate along the throat and should also be present in the transition region. The author thanks Ofer Aharony for explaining this point.

Initially we will start with the simplest situation where p and q are both small so that we can neglect the back reaction of the probe on the geometry. It is easy to identify the corresponding configuration in the dual field theory,

$$\langle\Phi_1\rangle = \alpha_1\left(I_{(q)}\otimes U_{(p)}\right)\oplus O_{(N-pq)} \qquad \langle\Phi_2\rangle = \alpha_2\left(I_{(q)}\otimes V_{(p)}\right)\oplus O_{(N-pq)} \tag{A.1}$$

and $\langle\Phi_3\rangle = 0$ and $O_{(k)}$ denotes the $k\times k$ matrix with zero entries. For general values of the deformation parameter, this is not a vacuum of the β-deformed theory and yields a non-zero classical potential energy,

$$V = \frac{4|\alpha_1|^2|\alpha_2|^2}{g^2}pq\left|\sin\left(\frac{\beta}{2}-\frac{\pi}{p}\right)\right|^2 . \tag{A.2}$$

The fact that the potential vanishes for $\beta = 2\pi/p$ for all values of α_1 and α_2 indicates that, for this value of the deformation parameter, the β-deformed theory has a Higgs branch on which the gauge group is broken as,

$$G = \mathrm{U}(N)\to\mathrm{U}(q)\times\mathrm{U}(N-pq) . \tag{A.3}$$

As we are interested in small values of the deformation parameter we will work with $1\ll p\ll N$ and restrict our attention to the case $|\beta|\sim 1/p\ll 1$. In this case the potential simplifies and becomes,

$$V = \frac{4|\alpha_1|^2|\alpha_2|^2}{g^2}pq\left|\frac{\beta}{2}-\frac{\pi}{p}\right|^2 = \frac{4|\alpha_1|^2|\alpha_2|^2}{g^2}\left(\frac{q\pi^2}{p}-q\pi\mathrm{Re}[\beta]+\frac{pq|\beta|^2}{4}\right). \tag{A.4}$$

We will now show how the first two terms in this calculation can be reproduced (up to numerical constants) in supergravity.

The relevant terms in the Dirac-Born-Infeld action of the probe brane are,

$$S_{\mathrm{DBI}} = -\frac{\mu_5}{g_s}\int d^6\xi\left[-\det\left(G_{||}\right)\det\left(g_s^{-\frac{1}{2}}e^{\frac{\phi}{2}}G_\perp + 2\pi\alpha'\mathcal{F}\right)\right]^{\frac{1}{2}} + \mu_5\int\left(C_6 + 2\pi\alpha'\mathcal{F}\wedge C_4\right) \tag{A.5}$$

where $G_{||}$ and $G_\perp$ are the pullback metric in the $R^{3,1}$ and T^2 components of the D5 worldvolume respectively and $\mathcal{F}$ is defined in (6.4) above. We choose coordinates $\xi_A = x_\mu$ for $A=\mu=0,1,2,3$ along $R^{3,1}$ and $\xi_4=\chi_1$, $\xi_5=\chi_2$ on T^2 (where χ_1 and χ_2 are defined below equation (6.5) in Subsection 6.1). This gives,

$$\det G_{||} = \mathcal{Z}^{-2} \qquad \det G_\perp = \mathcal{Z} . \tag{A.6}$$

As we have p units of D3 brane charge, the flux of $\mathcal{F}$ is quantized according to,

$$\int_{T^2}\mathcal{F} = 2\pi p . \tag{A.7}$$

Writing $\mathcal{F} = \mathcal{F}_{12}d\chi_1\wedge d\chi_2$, a uniform flux density then implies that $\mathcal{F}_{12} = p/2\pi r_1 r_2$. One can check that,

$$\frac{4\pi^2\alpha'^2\mathrm{det}\mathcal{F}}{\mathrm{det}G_\perp} = \frac{\alpha'^2p^2}{\mathcal{Z}r_1^2r_2^2} . \tag{A.8}$$

This quantity is the ratio of the two terms appearing inside the determinant in the first term in the DBI action (A.5). It is proportional to the ratio σ_3^2/σ_5^2 where σ_3 and σ_5 are the energy density induced by the D3 brane and D5 brane charge respectively. If we choose AdS warp factor $\mathcal{Z}_{\mathrm{UV}} = \mathcal{R}^4/r^4$ we find,

$$\frac{\sigma_3^2}{\sigma_5^2} \sim \frac{p^2}{g_s N}\,\frac{r^4}{r_1^2 r_2^2}\,. \tag{A.9}$$

Thus for a probe brane located at $r \sim \bar{r}$, we find that $\sigma_3 \gg \sigma_5$ provided $g_s N \ll p^2$. In this case the determinant in the DBI action (A.5) can be expanded as,

$$\sqrt{\det\left(g_s^{-\frac{1}{2}} e^{\frac{\phi}{2}} G_\perp + 2\pi\alpha' \mathcal{F}\right)} \simeq 2\pi\alpha' \sqrt{\det \mathcal{F}} + \frac{\det G_\perp}{4\pi\alpha' \sqrt{\det \mathcal{F}}}\,. \tag{A.10}$$

This expansion reflects the dominance of D3 brane charge over D5 charge in our chosen regime of parameters. In an approximation where the D5 brane charge is neglected completely, we know that the potential experienced by the probe should vanish reflecting the vanshing forces between D3 branes. Correspondingly, the first term on the right-hand side of (A.10) is exactly cancelled by the contribution of $C_{(4)}$ to the Chern-Simons term in the DBI action (A.5). In the absence of a non-zero backgound for $C_{(6)}$ the first non-cancelling term comes from the second term in (A.10). The resulting contribution to the potential energy of the configuration is,

$$V_0 = \frac{\mu_5}{g_s}\int_{T^2} d^2\xi \frac{\sqrt{\det G_\parallel}\det G_\perp}{4\pi\alpha'\sqrt{\det \mathcal{F}}} = \frac{2\pi^2\mu_5}{\alpha'}\,\frac{q r_1^2 r_2^2}{g_s p}\,. \tag{A.11}$$

Note that all dependence on the warp factor $\mathcal{Z}$ has vanished. Bearing in mind the identifications $r_1 = |\alpha_1|(2\pi\alpha')$ and $r_2 = |\alpha_2|(2\pi\alpha')$ we can evaluate V_0 in terms of the dual field theory variables giving,

$$V_0 \sim \frac{q|\alpha_1|^2|\alpha_2|^2}{g^2 p}\,. \tag{A.12}$$

This matches the first term on the right-hand side of the field theory potential up to numerical constants.

We now introduce a small non-zero background for the complex threeform field strength.

$$G^+_{(3)} = G_{(3)} + i \star_6 G_{(3)} = \left(\frac{i\rho\beta}{3g_s}\right)\mathcal{Z}(r)d\omega_2 \tag{A.13}$$

with,

$$\omega_2 = z_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + z_2 z_3 d\bar{z}_2 \wedge d\bar{z}_3 + z_3 z_1 d\bar{z}_3 \wedge d\bar{z}_1\,. \tag{A.14}$$

This result in a non-zero value of the dual six form potential. In the geometry (6.7) the relevant equation of motion takes the form,

$$dB_{(6)} - \tau dC_{(6)} = \frac{1}{g_s\mathcal{Z}} G^+_{(3)} \wedge dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \tag{A.15}$$

which yields

$$C_{(6)} = -\frac{\rho}{3g_s}\mathrm{Re}[\beta\omega_2] \wedge dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3\,. \tag{A.16}$$

JHEP07(2004)016

The six-form potential (A.16) contributes to the potential energy density via the $C_{(6)}$ coupling in the DBI action (A.5). The resulting contribution is,

$$V_1 = -\mu_5 \frac{4\pi^2\rho}{9} \frac{q r_1^2 r_2^2}{g_s} \mathrm{Re}[\beta] \,. \tag{A.17}$$

Once again this does not depend on the corresponding warp factor $\mathcal{Z}$. In field theory variables we have,

$$V_1 \sim \frac{q|\alpha_1|^2|\alpha_2|^2}{g^2} \mathrm{Re}[\beta] \,. \tag{A.18}$$

This matches the linear term in the field theory potential (A.4) up to numerical constants. This is already enough to see that the contribution V_1 of the background flux will cancel V_0, and thereby stabilize the probe brane at an arbitrary radial position, for an isolated value of the deformation parameter $\mathrm{Re}[\beta]$. A more complete calculation would involve determining the term of order $|\beta|^2$ and fixing the numerical constants as in the $\mathcal{N} = 1^*$ case of Polchinski and Strassler.

The next step, following [15], is to relax the condition $p \ll N$. The corresponding back-reaction of the D3 brane charge carried by the probe can be accounted for by modifying the warp factor $\mathcal{Z}$ appearing in the background geometry (6.7). Provided the expansion (A.10) still holds, the calculation goes through exactly as before. In particular the leading terms in the potential experienced by the probe do not depend on $\mathcal{Z}$ and our results are unchanged. Next we can go to the the case of the Higgs branch vacuum discussed in the text which corresponds to $p = n$ and $q = m$ where $N = mn$. In this case the corresponding warp factor is given in (6.9). To check that the probe calculation continues to make sense, we must check the condition (A.8) for the validity of the expansion (A.10). Using the UV form of the metric $\mathcal{Z}_{UV} = \mathcal{R}^4/r^4$ for a brane located at $r = \bar{r}$ the condition is satisfied provided $g_s m \ll n$. This condition is the expected one for D3 brane charge to dominate over D5 brane charge.

As long as the condition $g_s m \ll n$ is satisfied the calculation of the potential given above is unchanged. The brane configuration is stable for a single value of $\beta \sim 1/n$ in accordance with field theory expectations. A more complete calculation could check that the relevant value is actually $\beta = 2\pi/n$.

B. Deconstructing DSLST

In this appendix we fill in some of the details required to adapt the deconstruction proposal of section 5 to the double-scaled case. As for the theory at the origin it is convenient to first identify the relevant limit of the Higgs branch theory and then use S-duality to reinterpret the limit in terms of the confining phase theory.

We begin by considering the Higgs phase of the U(N) β-deformed theory with $N = mn$ and $\beta = 2\pi/n$. We will now consider a particular vacuum of the theory where the gauge symmetry is broken to U(1)^m. In particular we choose the scalar VEVs as,

$$\langle\Phi_1\rangle = \alpha_1 I_{(m)} \otimes U_{(n)} \qquad \langle\Phi_2\rangle = \alpha_2 I_{(m)} \otimes V_{(n)} \qquad \langle\Phi_3\rangle = \alpha_3 U_{(m)} \otimes V^\dagger_{(n)} U^\dagger_{(n)} \tag{B.1}$$

where α_1, α_2 and α_3 are complex numbers. We can also describe this vacuum in terms of the independent gauge-invariant chiral operators which are non-zero. These are,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_1^{rn}\right]\right\rangle = \alpha_1^{rn} \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_2^{sn}\right]\right\rangle = \alpha_2^{sn} \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_3^{N}\right]\right\rangle = (-1)^{N-1}\alpha_3^N \tag{B.2}$$

where $r, s = 1, 2, \ldots, m$ are integers. To absorb an irrelevant phase we define $\mu = \exp(\pi i(N-1)/N)\alpha_3$. Fixing these VEVs uniquely specifies the vacuum state. However, in this vacuum there are also an additional set of non-vanishing chiral operators,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Phi_1^{km}\Phi_2^{km}\Phi_3^{km}\right]\right\rangle = \exp(i\nu_k)\alpha_1^{km}\alpha_2^{km}\mu_3^{km} \tag{B.3}$$

for integer $k = 1, 2, \ldots, n$, where $\exp(i\nu_k)$ is an unimportant overall phase.

In addition to breaking the low energy gauge group down to $\mathrm{U}(1)^m$, the scalar VEVs also break the $\mathrm{U}(1)^3$ R-symmetry of the β-deformed theory down to $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_m$. The massless fields correspond to m vector multiplets of $\mathcal{N} = 4$ supersymmetry. At finite n, the theory also contains a complicated spectrum of massive excitations. If we choose $|\mu| \ll |\alpha_1|$, $|\alpha_2|$ then the lightest W-bosons have masses,

$$M_W^{(ab)} = 2|\mu| \sin\left(\frac{\pi}{m}|a-b|\right) \tag{B.4}$$

where $a, b = 1, 2, \ldots, m$ are integers. Each of these states is BPS saturated with respect to the low-energy $\mathcal{N} = 4$ supersymmetry. Over each of these light states there are two towers of states corresponding to the Kaluza-Klein modes of a six-dimensional field compactified to four-dimensions on a torus.

At weak coupling, the string theory dual of this vacuum can be worked out by a simple modification of the analysis of the $\alpha_3 = 0$ case given in [9]. As before we have a configuration of N D3 branes polarised into m toroidally wrapped D5 branes. For $\alpha_3 \neq 0$, however each D5 brane is wrapped around a particular torus T_a^2 for $a = 1, 2, \ldots, m$. In terms of the complex coordinates (6.2) the torus T_a^2 is defined by $\rho_i = r_i = |\alpha_i|(2\pi\alpha')$ for $i = 1, 2, 3$. For the torus T_a^2 corresponding to the a'th wrapped D5 brane we also have the condition,

$$\psi_1 + \psi_2 + \psi_3 = \frac{2\pi a}{m} \tag{B.5}$$

for $a = 1, 2, \ldots, m$. The spectrum (B.4) of light W-bosons corresponds to that of the lightest fundamental strings stretched between the D5 branes for $r_3 \ll r_1$, r_2.

For large values of the gauge coupling we can perform an S-duality transformation to the dual confining phase theory with coupling $\tilde{g}^2$ and deformation parameter $\tilde{\beta} = 8\pi i/\tilde{g}^2 n$. The corresponding gauge invariant vacuum expectation values are,

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_1^{rn}\right]\right\rangle = \tilde{\alpha}_1^{rn} \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_2^{sn}\right]\right\rangle = \tilde{\alpha}_2^{sn} \qquad \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\tilde{\Phi}_3^{N}\right]\right\rangle = \tilde{\mu}^N \tag{B.6}$$

with $\tilde{\alpha}_i = (\tilde{g}^2/4\pi)\alpha_i$ for $i = 1, 2$ and $\tilde{\mu} = (\tilde{g}^2/4\pi)\mu$. The string dual now consists of m NS5 branes wrapped on torii T_a^2 for $a = 1, 2, \ldots, m$. The spectrum of light W-bosons corresponds to D-strings stretched between the NS5 branes with masses,

$$M_W^{(ab)} = 2|\tilde{\mu}|\left(\frac{4\pi}{\tilde{g}^2}\right)\sin\left(\frac{\pi}{m}|a-b|\right). \tag{B.7}$$

JHEP07(2004)016

At strong coupling, $\tilde{g}^2 n \gg m$, we can find the string dual of the vacuum state described above by the methods used in the text. The result is again m NS5 branes wrapped on S^5 in a geometry of the form (6.7). The NS5's are now separated in the angular directions of S^5. Importantly, we are interested in the case where the typical separation, r_0, is small compared with the thickness of the shell where the supergravity solution is well approximated by the near-horizon geometry of m flat NS5 branes. In this regime the deformation is equivalent to distributing these flat NS5 branes around a circle of radius r_0 in their four transverse directions

Finally we take the continuum limit, Limit I as before. Thus we take $n \to \infty$ and $\tilde{g}^2 \to 0$ with $\tilde{g}^2 n$, m $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ fixed. The only new feature is that we simultaneously scale $\tilde{\mu}$ so as to hold the W-boson masses (B.7) fixed. Thus we simultaneously take the limit $|\tilde{\mu}| \to 0$ with $|\tilde{\mu}|n$ fixed. The result of this scaling is DSLST with parameters,

$$M_s = \sqrt{\frac{|\tilde{\alpha}_1||\tilde{\alpha}_2|}{\tilde{g}^2 n}} \qquad \bar{M} = \frac{2\pi|\tilde{\alpha}_1||\tilde{\alpha}_2|}{|\tilde{\mu}|n}\,. \tag{B.8}$$

References

[1] N. Seiberg, *New theories in six dimensions and matrix description of M-theory on T^5 and $T^5/\mathbb{Z}_2$*, *Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221];
M. Berkooz, M. Rozali and N. Seiberg, *On transverse fivebranes in m(atrix) theory on T^5*, *Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[2] O. Aharony, *A brief review of 'little string theories'*, *Class. and Quant. Grav.* **17** (2000) 929 [hep-th/9911147].

[3] D. Kutasov, *Introduction to little string theory*, prepared for *ICTP spring school on superstrings and related matters*, Trieste, Italy, 2-10 Apr 2001.

[4] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 216.

[5] N. Arkani-Hamed, A.G. Cohen and H. Georgi, *(De)constructing dimensions*, *Phys. Rev. Lett.* **86** (2001) 4757 [hep-th/0104005].

[6] M.B. Halpern and W. Siegel, *Electromagnetism as a strong interaction*, *Phys. Rev.* **D 11** (1975) 2967;
O.J. Ganor, *Six-dimensional tensionless strings in the large-N limit*, *Nucl. Phys.* **B 489** (1997) 95 [hep-th/9605201];
O.J. Ganor and S. Sethi, *New perspectives on Yang-Mills theories with sixteen supersymmetries*, *J. High Energy Phys.* **01** (1998) 007 [hep-th/9712071];
I. Rothstein and W. Skiba, *Mother moose: generating extra dimensions from simple groups at large-N*, *Phys. Rev.* **D 65** (2002) 065002 [hep-th/0109175].

[7] N. Arkani-Hamed, A.G. Cohen, D.B. Kaplan, A. Karch and L. Motl, *Deconstructing (2,0) and little string theories*, *J. High Energy Phys.* **01** (2003) 083 [hep-th/0110146].

[8] A. Adams and M. Fabinger, *Deconstructing noncommutativity with a giant fuzzy moose*, *J. High Energy Phys.* **04** (2002) 006 [hep-th/0111079].

[9] N. Dorey, *S-duality, deconstruction and confinement for a marginal deformation of $N = 4$ SUSY Yang-Mills*, hep-th/0310117.

[10] J. Nishimura, S.-J. Rey and F. Sugino, *Supersymmetry on the noncommutative lattice*, *J. High Energy Phys.* **02** (2003) 032 [hep-lat/0301025].

[11] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[12] D. Kabat and I.W. Taylor, *Spherical membranes in matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 181 [hep-th/9711078];
I.W. Taylor, *The m(atrix) model of M-theory*, hep-th/0002016.

[13] S.-J. Rey, *Gravitating m(atrix) Q-balls*, hep-th/9711081.

[14] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[15] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[16] M.R. Douglas, *D-branes and discrete torsion*, hep-th/9807235;
M.R. Douglas and B. Fiol, *D-branes and discrete torsion, II*, hep-th/9903031.

[17] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[18] D. Berenstein and R.G. Leigh, *Discrete torsion, AdS/CFT and duality*, *J. High Energy Phys.* **01** (2000) 038 [hep-th/0001055];
D. Berenstein, V. Jejjala and R.G. Leigh, *Marginal and relevant deformations of $N = 4$ field theories and non-commutative moduli spaces of vacua*, *Nucl. Phys.* **B 589** (2000) 196 [hep-th/0005087]; *Noncommutative moduli spaces and t duality*, *Phys. Lett.* **B 493** (2000) 162 [hep-th/0006168].

[19] N. Dorey, T.J. Hollowood and S.P. Kumar, *S-duality of the Leigh-Strassler deformation via matrix models*, *J. High Energy Phys.* **12** (2002) 003 [hep-th/0210239].

[20] K.A. Intriligator, *Bonus symmetries of $N = 4$ super-Yang-Mills correlation functions via AdS duality*, *Nucl. Phys.* **B 551** (1999) 575 [hep-th/9811047].

[21] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[22] M. Grana and J. Polchinski, *Supersymmetric three-form flux perturbations on AdS_5*, *Phys. Rev.* **D 63** (2001) 026001 [hep-th/0009211].

[23] O. Aharony, B. Kol and S. Yankielowicz, *On exactly marginal deformations of $N = 4$ sym and type-IIB supergravity on $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 039 [hep-th/0205090].

[24] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [hep-th/9809022].

[25] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[26] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110]; *Comments on double scaled little string theory*, *J. High Energy Phys.* **01** (2000) 023 [hep-th/9911039].

[27] O. Aharony, A. Giveon and D. Kutasov, *LSZ in LST*, hep-th/0404016.

[28] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-N noncommutative field theories*, *J. High Energy Phys.* **11** (1999) 007 [hep-th/9909215].

[29] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[30] G. 't Hooft, *A property of electric and magnetic flux in nonabelian gauge theories*, *Nucl. Phys.* **B 153** (1979) 141.

[31] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[32] J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Finite n matrix models of noncommutative gauge theory*, *J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041]; *Nonperturbative dynamics of noncommutative gauge theory*, *Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Lattice gauge fields and discrete noncommutative Yang-Mills theory*, *J. High Energy Phys.* **05** (2000) 023 [hep-th/0004147].

[33] T. Banks, N. Seiberg and S.H. Shenker, *Branes from matrices*, *Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157];
M. Li, *Strings from IIB matrices*, *Nucl. Phys.* **B 499** (1997) 149 [hep-th/9612222];
H. Aoki et al., *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys.* **B 565** (2000) 176 [hep-th/9908141];
I. Bars and D. Minic, *Non-commutative geometry on a discrete periodic lattice and gauge theory*, *Phys. Rev.* **D 62** (2000) 105018 [hep-th/9910091];
Y. Kimura, *Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model*, *Prog. Theor. Phys.* **106** (2001) 445 [hep-th/0103192];
D. Bigatti, *Gauge theory on the fuzzy torus*, hep-th/0109018.

[34] N. Seiberg, *A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation*, *J. High Energy Phys.* **09** (2000) 003 [hep-th/0008013].

[35] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[36] I.W. Taylor and M. Van Raamsdonk, *Multiple Dp-branes in weak background fields*, *Nucl. Phys.* **B 573** (2000) 703 [hep-th/9910052].

[37] P. Kraus, F. Larsen and S.P. Trivedi, *The coulomb branch of gauge theory from rotating branes*, *J. High Energy Phys.* **03** (1999) 003 [hep-th/9811120].

[38] K. Sfetsos, *Branes for Higgs phases and exact conformal field theories*, *J. High Energy Phys.* **01** (1999) 015 [hep-th/9811167].

JHEP07(2004)016